Optimal Biased Design of Dynamic Multi-Battle Team Contests *

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Abstract

We study the optimal biased design of dynamic multi-battle team contests, in which two asymmetric teams compete over a series of battles with a majoritarian objective. The designer can impose a treatment to bias each battle contest in order to maximize the expected aggregate effort. With full homogeneity, the optimal bias fully balances each battle contest, i.e., players are equally likely to win a battle, regardless of the previous outcomes. By introducing outcome-dependent heterogeneity, the history independence result collapses. Nevertheless, we propose a general program whose solution yields the optimal outcome-dependent biases. In this case, we show that the full-balance rule is no longer effort-maximizing generically, even when all players are fully homogeneous. This indicates that outcome-dependent heterogeneity plays a crucial role in determining the optimal biases of dynamic contests.

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1 Introduction

Sequential multi-battle contests are commonly observed in reality and have been extensively studied in the literature.[1] Some of the studies focus on multi-battle individual contests, in which the same two players compete against each other over a series of confrontations.[2] While others consider the setting of multi-battle team contests, in which players from two teams form pairwise matches to compete head-to-head in their own battles.[3] In this paper, we examine how an effort-maximizing designer optimally biases a dynamic majoritarian multi-battle team contest, which is a natural counterpart of the work by Barbieri and Serena (2022) who analyze the effort-maximizing biases in a dynamic multi-battle contest between two individuals.

It is well understood that, in a single-round individual contest, the contest designer can level the playing field to fuel the competition by favoring the weaker. However, to what extent this insight can be extended to a *dynamic* contest between asymmetric teams remains less clear.[4] In this paper, we attempt to shed light on the general question by first considering a dynamic multi-battle team contest. In such a contest, two teams with asymmetric valuations compete over N pairwise battles sequentially under the best-of-N rule, where N can be any positive odd number. The designer can influence contestants' relative competitiveness in *battles* by imposing the multiplicative biases to rescale contestants' effort entries. Our research question is how an effort-maximizing designer optimally biases such a dynamic multi-battle contest between two asymmetric teams.

A key feature of a dynamic setting is the observability of the history. As the contest unfolds, previous outcomes become known to the public, which naturally leads to a greater number of contingencies that must be taken into account. To capture this feature, we con-

[1] For instance, in legislative elections, candidates from two major parties compete for legislative seats and the party with a majority of the seats gains the control of legislature, e.g., Republicans and Democrats in an election for the House of Representatives (see, e.g., Snyder (1989) and Klumpp and Polborn (2006)); In sports between teams, the best-of-N rule is prevalent, i.e., the final winner is the one accumulating majority battle wins, such as tennis matches and the National Basketball Association's Finals playoff series (see, e.g., Ferrall and Smith (1999) and Malueg and Yates (2010)); In parent races, opponent research teams compete and a team can patent an invention that consists of sufficient technological breakthroughs (see, e.g., Harris and Vickers (1987)), among others.

[2] See, for instance, Konrad and Kovenock (2009), Feng and Lu (2018), and Klumpp, Konrad, and Solomon (2019).

[3] See, for example, Fu, Lu, and Pan (2015), Häfner (2017), Barbieri and Serena (2019), and Feng, Jiao, Kuang, and Lu (2023).

[4] In the literature review, we will provide a more detailed discussion of studies on optimal biases in dynamic contests.

sider both homogeneous and heterogeneous cases. For homogeneous case, both the contest technology and the choice of bias remain the same, regardless of the revealed outcomes. We further allow that the contest technology and/or the bias can be contingent on the previous battle outcomes, which we refer to as *outcome-dependent heterogeneity.* For the sake of generality, we adopt a generalized Tullock contest technology to model each battle, i.e., the discriminatory power $r \in [0, +\infty)$. A question naturally arises here: Whether the conventional wisdom of levelling the playing field can be extended to such a dynamic contest between two asymmetric teams in the presence of homogeneity/heterogeneity.

In our model, two teams containing an equal number of players compete over N pairwise battles sequentially: each player of one team fights against his opponent from the rival team in a battle, as a result, each player bears the effort cost of that battle only. Players are homogeneous within a team in terms of their marginal effort costs. As battles are played sequentially, players can observe the outcomes of previous battles, i.e., the state of contest (n_1, n_2), where n_i denotes the number of battle wins secured by team i. A best-of-N rule is applied to determine the winning team, that is, the team accumulating a majority of battle wins is rewarded with the final trophy. We assume that players within the same team value the team's win equally, which is equivalent to assume that the team's prize would be divided equally among its players. Our model is inherited from Fu, Lu, and Pan (2015) but differ from their setting in two aspects. First, we allow asymmetric teams, i.e., two teams can value the final trophy differently. Second, this outcome-dependent heterogeneity is state-specific, which directly causes the failure of history independence result.[5]

In an one-shot contest, the total effort put forth by players is determined by the their valuations of winning, the contest technology, as well as the bias chosen. To analyze a dynamic contest, we propose a recursive formula to track players' incentives, i.e., their prize spreads. As a by-product, we extend a key observation of Fu, Lu, and Pan (2015) into an asymmetric contest between two teams. That is, despite the paired players having different prize spreads, they always have a common adjusted prize spread, regardless of the state of the contest.[6] We then apply the recursive approach to determine the adjusted prize spreads across the states, which enables us to characterize the expected total effort functions for homogeneous/heterogeneous cases. On that basis, we derive the optimal biases in both cases. In particular, we find that a player's prize spread at a state is completely determined

[5] We will also discuss the differences in the literature review.

[6] Specifically, the adjusted prize spread equals a player's prize spread divided by his team's valuation of winning the whole contest.

by the winning probabilities at attainable states in subsequent battles.

With homogeneous battles, the optimal bias completely counterbalances the asymmetry between the paired players that is inherited from their teams. Put differently, the full-balance rule—two paired players always have equal chance to win a battle—is effort-maximizing, regardless of the previous outcomes and players' prize spreads. This result echoes the existing findings in the literature on biased design in one-shot contests: A level playing field would maximize the competition. Our analysis shows that even though there is no loss to focus on a single battle to characterize the optimal bias, the intuition is different: The optimal bias not only intensifies the competition in each battle at a given state, but also maximizes the likelihood of attaining a balanced state, e.g., (k, k) in a contest with $N = 2k + 1$ battles.

With outcome-dependent heterogeneity (i.e., contest technologies and/or biases can vary with the revealed battle outcomes), it is worth to noting that history-independence result collapses.[7] As a consequence, the conventional approach to compute total effort is no longer applicable. To derive total effort in this case, we trace each possible history by introducing two concepts, path and adjusted path. Path is utilized to compute the probability of attaining a state, whereas adjust path is used to determine players' incentives at possible states. After characterizing the total effort function, we formulate a general multi-dimensional program whose solution yields the optimal biases. This approach offers a practical method for computing the optimal design numerically. We further simplify the general program into a system of single-dimensional subproblems, which enables us to establish several useful properties about the optimal biases. In particular, we prove that the full-balance rule is no longer effort-maximizing generically in the presence of the outcome-dependent heterogeneity. In other words, by introducing state-specific heterogeneity (e.g., the contest technology varies with the state), players in general do not have equal chance to win a battle under the optimal design, even all players are fully homogeneous in terms of their marginal effort costs.[8] Compared to the homogeneous case, the result implies that state-specific heterogeneity could be a key factor in determining the optimal biases.

In dynamic team contests with outcome-dependent heterogeneity, the optimal biases

[7] History-independence result means that the outcomes of past battles do not distort the winning probabilities in future battles. In particular, it requires that a team's equilibrium probability of winning a battle remains a constant regardless of the revealed state. However, when r varies with the state (n_1, n_2) and/or α is allowed to be contingent on (n_1, n_2), the winning probability depends on both r and α, and therefore the state.

[8] We sometimes also call outcome-dependent heterogeneity as state-specific heterogeneity, since state is defined as the previous battle outcomes in our model.

should not only counterbalance the asymmetry between players, but also manage this heterogeneity over the states. Our result reveals that there is a trade-off in achieving the two goals, since full-balance rule completely eliminates the asymmetry of each battle contest but remains suboptimal in eliciting efforts in the presence of the outcome-dependent heterogeneity. This further indicates that outcome-dependent heterogeneity that is not displayed in an one-shot contest could be crucial in affecting the optimal biased design of a dynamic contest.

A 3-battle Example: To see how this outcome-dependent heterogeneity impacts on the design of the optimal biases, consider a 3-battle contest between two symmetric teams, assuming that players are fully homogeneous. We introduce outcome-dependent heterogeneity in the following way: The second battle would be an all-pay contest if team A wins a battle; a lottery contest is applied for all other remaining states/battles. Equivalently, $r(1,0) = +\infty$ and $r(0,0) = r(0,1) = r(1,1) = 1$. In this example, it is optimal to favor team A tremendously in the first battle (i.e., $\alpha(0,0) = +\infty$) so that the contest would certainly move to an all-pay contest in the second battle, in which two players would be treated equally, i.e., $\alpha(1,0) = 1$. To further intensify the competition, the designer should favor team B as much as possible in third battle when there is a tie (i.e., $\alpha(1,1) = 0$), as it would incentivize team A in the second battle, which is an all-pay contest. Applying the aforementioned biases ($\alpha(0,0) = +\infty$, $\alpha(1,0) = 1$, $\alpha(1,1) = 0$, and $\alpha(0,1)$ can be any positive value), the team contest boils down to an individual contest wherein two players compete against each other in a symmetric all-pay contest, which yields the greatest effort. More generally, we resort to the general program (12) to search for the optimal biases.

Related literature: This paper belongs to the interaction of three streams of literature: dynamic contests, group contests, and the design of biased contests.

On dynamic contest, a stream of literature focuses on the competition between two individuals wherein the same two players sequentially interact in all battles, including Harris and Vickers (1987), Snyder (1989), Ferrall and Smith (1999), Klumpp and Polborn (2006), Konrad and Kovenock (2009), McFall, Knoeber, and Thurman (2009), Malueg and Yates (2010), Sela (2011), Gelder (2014), Gelder and Kovenock (2017), Klumpp, Konrad, and Solomon (2019), and Gauriot and Page (2019), among others.[9] Many of those works consider majoritarian multi-battle individual contests and identify the strategic momentum/discouragement effect in their settings, i.e., an initial lead incentivizes an early winner but discourages an early loser in subsequent battles. In particular, Klumpp, Konrad, and Solomon (2019) ex-

[9] Other papers focus on prize designs in dynamic contests between individual players, including Feng and Lu (2018); Jiang (2018); Sela and Tsahi (2020); and Clark and Nilssen (2020), among others.

amine the sequential majoritarian Blotto contest, in which the same two players sequentially interact in a series of battles with an arbitrary odd length. While our paper focuses on the sequential majoritarian team contests with arbitrary odd battles wherein players of two teams form pairwise matches to compete head-to-head in their own battles.

Our paper is also related to the literature on group/team contests. Many studies model the team competition by using an index to represent players' efforts within a team (e.g., aggregating players' efforts), including Baik, Kim, and Na (2001), Barbieri, Malueg, and Topolyan (2014), Topolyan (2014), Chowdhury, Lee, and Topolyan (2016), Eliaz and Wu (2018), Crutzen, Flamand, and Sahuguet (2020), Fu and Lu (2020), Arbatskaya and Konishi (2021), and Cubel and Sanchez-Pages (2022), etc.[10] While we consider a best-of-N team contest with pairwise battles, in which battles are played sequentially and the final trophy is determined by a team's number of battle wins. The multi-battle team contest is a natural counterpart of multi-battle individual contest. More importantly, the momentum/discouragement effect that prevails in dynamic multi-battle individual contests does not exist in multi-battle team contests. This allows us to better understand the role of the momentum/discouragement effect in affecting the optimal biases of dynamic contests.

Our team contest model is built on Fu, Lu, and Pan (2015) and analyzed later by Häfner (2017), Barbieri and Serena (2019), Feng, Jiao, Kuang, and Lu (2023) among others.[11] In terms of model setup, we differ from the aforementioned studies in two aspects. First, we allow asymmetric teams, as a result, two matched players no longer equally value the win of a battle.[12] Even though, we establish that paired players always have the same adjusted value of winning a battle.[13] Second, with outcome-dependent heterogeneity, the history independence result no longer holds. This is because when the biases and/or contest technology vary with the state of the contest, the observability of the history plays a role in affecting players' incentives.

The biased design has been extensively studied in the setting of static contests, see, e.g., Li and Yu (2012), Franke, Kanzow, Leininger, and Schwartz (2013), Drugov and Ryvkin (2017), Franke, Kanzow, Leininger, and Schwartz (2014), Franke, Leininger, and Wasser (2018), Fu

[10] Many of these works also assume that a team's trophy is a public good among its team players.

[11] Häfner (2017) considers a tug-of-war team contest, Barbieri and Serena (2019) examine the temporal structure by focusing on the winner's effort, and Feng, Jiao, Kuang, and Lu (2023) analyse the optimal prize allocation rule.

[12] As a consequence, Theorem 1 of Fu, Lu, and Pan (2015) is no longer applicable in our model.

[13] The adjusted value equals a player's prize spread divided by his team's value of winning the whole contest. We will formally introduce the concept in Definition 5.

and Wu (2020), Deng, Fu, and Wu (2021) among others.[14] Franke, Leininger, and Wasser (2018) and Fu and Wu (2020) consider both additive headstarts and multiplicative biases in static contests with multiple individual players. In particular, Fu and Wu (2020) show that the contest designer does not benefit from giving headstarts to contestants. In contrast to these works, we consider a dynamic setting.

There also exist studies concerning biases in dynamic contests, including Meyer (1991), Meyer (1992), Ederer (2010), etc.[15] For instance, Meyer (1991) consider a multi-stage setting wherein two players compete for a single prize and establishes the result of "favor-the-lead", i.e., the optimal final-period bias is in the favor of the leader. In a two-stage setting, Clark, Nilssen, and Sand (2012), Möller (2012), Beviá and Corchón (2013), Esteve-González (2016), and Klein and Schmutzler (2017) consider a competition between two individuals in which the first-stage winner is awarded an advantage status in the second-stage competition.[16] Fu and Wu (2022) consider a two-stage elimination contest, in which the organizer can bias the second-stage competition based on finalists' interim rankings. Barbieri and Serena (2022) study a best-of-three contest between two ex-ante symmetric individuals and consider both the victory-dependent biases and victory-independent biases, which are called outcome-dependent and outcome-independent biases in our context, respectively.[17] We differ from these studies by considering dynamic team contests with pairwise battles. The convenience of the setting allow us to consider arbitrary odd battles, generalized Tullock contest technologies, and outcome-dependent heterogeneity, i.e., the biases and/or the contest technologies can vary with the state of the contest.

Our work complements Barbieri and Serena (2022). They study the optimal biases of a dynamic multi-battle individual contest wherein the same two ex ante symmetric players compete against each other in all battles.[18] An important feature of multi-battle individual contests is the momentum/discouragement effect, which is absent in multi-battle team con-

[14] While many other studies focus on the (additive) headstart, e.g., Clark and Riis (2000), Konrad (2002), Siegel (2009), Kirkegaard (2012), and Siegel (2014).

[15] Ederer (2010) examines the feedback policy in a two-stage contest and consider one case wherein the agent's production function takes the multiplicative form.

[16] Ridlon and Shin (2013) show that if employees' abilities are sufficiently different, favoring the first-period loser in the second period increases the total effort over both periods. However, if abilities are sufficiently similar, total effort increases the most in response to a handicapping strategy of favoring the first-period winner.

[17] Feng and Lu (2018) focus on the optimal prize allocation in a best-of-three individual contest.

[18] Barbieri and Serena (2022) additionally consider winner's effort maximization, while we focus on total effort maximization by allowing arbitrary odd battles and outcome-dependent heterogeneity.

tests.[19] Comparing to Barbieri and Serena (2022), we further confirm that the momentum effect plays a role in designing the effort-maximizing biases in dynamic contests: Barbieri and Serena (2022) show that the optimal victory-independent biases do not leave players equally likely to win the overall contest, due to momentum effect. In contrast, with homogeneous battles, we prove that two matched players always have the same chance to win each battle under the optimal design, and therefore two teams are equally likely to win the whole contest. Moreover, by allowing the biases and/or the contest technologies vary with the state of the contest, we find that this state-specific heterogeneity could also be a key factor in determining the optimal biases. This kind of heterogeneity is not displayed in static contests and many dynamic contests with homogeneous battles.

The remainder of this paper is organized as follows. In Section 2, we introduce the basic contest model. In Section 3, we study an one-shot contest and consider 3-battle and 5-battle contests as examples. In Section 4, we analyze a general dynamic contest between two teams with $N = 2k + 1$ pairwise battles. Specifically, in Section 4.1, we characterize the effort-maximizing bias explicitly for homogeneous case. In Section 4.2, when outcome-dependent heterogeneity is introduced, we offer a general program to search for the optimal vector of biases. On that basis, we establish several useful properties of the effort-maximizing biases. Section 5 provides discussions and concludes.

2 Model

Consider two teams, indexed by A and B, compete in a contest with $N = 2k+1$ pairwise battles, where $k \in Z^+$. Each team consists of $2k + 1$ risk-neutral players. Players from opposing teams are paired up and compete against each other in their own individual battles. This means that each player only participates and fights in his own battle. We denote player $A(t)$ (resp. $B(t)$) as the one assigned by team A (resp. B) in battle $t \in \{1, ..., 2k + 1\}$. The overall outcome of the whole contest is determined by the results of individual battles through majority rule, i.e., a team wins the final trophy if and only if it secures at least $k+1$ battle wins.[20]

Denote team A's valuation (B's valuation) from winning the whole contest by v_A (resp.

[19] It is shown in Fu, Lu, and Pan (2015) that the momentum/discouragement effect is absent in a multi-battle team contest with symmetric teams.

[20] The majority rule is popular in both theory and practice, e.g., Klumpp and Polborn (2006), Klumpp, Konrad, and Solomon (2019), Barbieri and Serena (2022), and Feng, Jiao, Kuang, and Lu (2023), etc.

v_B). Throughout the paper, we assume that the team prize is a public good, i.e., all players within the same team value the team prize equally. It is equivalent to assume that the team prize will be divided equally among its team players. We model each component battle as a Tullock contest with an arbitrary $r \in (0, +\infty]$.[21]

We assume that the $2k+1$ disjoint battlefields are played successively. To track the history of battle outcomes, we denote (n_1, n_2) the state of contest, where n_1 is the number of battle wins secured by team A and n_2 is the number of battle wins secured by team B. At state (n_1, n_2), player $A(t)$'s (resp. $B(t)$) valuation of winning battle t is $\Delta u_A(n_1, n_2)$ (resp. $\Delta u_B(n_1, n_2)$), where $t = n_1 + n_2 + 1$.

3 Analysis

We begin our analysis by considering a battle contest. In an arbitrary battle t, two players $A(t)$ and $B(t)$ compete in a biased Tullock contest. Specifically, player $A(t)$'s winning probability is given by $p_{A(t)} = \alpha x^r_{A(t)} / \left(\alpha x^r_{A(t)} + x^r_{B(t)}\right)$ if at least one player spends positive effort. Player $A(t)$'s (resp. $B(t)$) valuation of winning battle t is Δu_A (resp. Δu_B). We assume unity effort cost. Player $A(t)$ chooses effort $x_{A(t)}$ to maximize

$$\frac{\alpha x^r_{A(t)}}{\alpha x^r_{A(t)} + x^r_{B(t)}} \Delta u_A - x_{A(t)}.$$

Likewise, player $B(t)$ chooses effort $x_{B(t)}$ to maximize

$$\frac{x^r_{B(t)}}{\alpha x^r_{A(t)} + x^r_{B(t)}} \Delta u_B - x_{B(t)}.$$

Suppose that $\Delta u_{A(t)}/v_A = \Delta u_{B(t)}/v_B$, where v_A and v_B are exogenously given.[22] We denote

$$\Delta u := \Delta u_A/v_A = \Delta u_B/v_B.$$

In the following property, we show that the sum of the expected efforts in battle t is always proportional to the adjusted valuation of winning battle t, Δu.

[21] We will allow r to be contingent on the state in our general analysis.

[22] We establish that $\Delta u_{A(t)}/v_A = \Delta u_{B(t)}/v_B$ in dynamic team contests with pairwise battles in Lemma 3.

Property 1. *In battle t, the expected effort from both players equals*

$$E\left[x_A + x_B\right] = \beta\left(v_A, v_B, \alpha, r\right)\Delta u, \tag{1}$$

where $\beta\left(v_A, v_B, \alpha, r\right)$ only depends on v_A, v_B, α, r, rather than the players' prize spreads, Δu_A and Δu_B.

Proof. To prove the property, we derive the players' equilibrium effort. To do so, we transform the biased Tullock contest into a non-biased one by adjusting player $A(t)$'s valuation. The equilibrium strategies in the later game are summarized in Lemma 1 in Feng and Lu (2018). The proof details and the analytical derivation of $\beta\left(v_A, v_B, \alpha, r\right)$ are provided in the appendix. □

For $r \in (0, +\infty)$, when $\alpha^{1/r}\Delta u_A \geq \Delta u_B$, i.e., $\alpha^{1/r} v_A \geq v_B$,

$$\beta\left(v_A, v_B, \alpha, r\right) = \begin{cases} \frac{r\alpha v_A^r v_B^r}{\left(\alpha v_A^r + v_B^r\right)^2}\left(v_A + v_B\right) & \text{if} \quad r \leq \widehat{r}\left(\frac{v_B}{\alpha^{1/r} v_A}\right), \\ \left(\frac{1}{r-1}\right)^{\frac{1}{r}}\left(1 - \frac{1}{r}\right)\frac{v_B}{\alpha^{1/r}}\left(1 + \frac{v_B}{v_A}\right) & \text{if} \quad r \in \left(\widehat{r}\left(\frac{v_B}{\alpha^{1/r} v_A}\right), 2\right], \\ \frac{1}{2\alpha^{1/r}}\left(1 + \frac{v_B}{v_A}\right) v_B & \text{if} \quad r \in (2, +\infty); \end{cases}$$

when $\Delta u_B > \alpha^{1/r}\Delta u_A$, i.e., $v_B > \alpha^{1/r} v_A$,

$$\beta\left(v_A, v_B, \alpha, r\right) = \begin{cases} \frac{r\alpha v_A^r v_B^r}{\left(\alpha v_A^r + v_B^r\right)^2}\left(v_A + v_B\right) & \text{if} \quad r \leq \widehat{r}\left(\frac{\alpha^{1/r} v_A}{v_B}\right), \\ \left(\frac{1}{r-1}\right)^{\frac{1}{r}}\left(1 - \frac{1}{r}\right)\alpha^{1/r}\left(1 + \frac{v_A}{v_B}\right) v_A & \text{if} \quad r \in \left(\widehat{r}\left(\frac{\alpha^{1/r} v_A}{v_B}\right), 2\right], \\ \frac{\alpha^{1/r}}{2}\left(1 + \frac{v_A}{v_B}\right) v_A & \text{if} \quad r \in (2, +\infty). \end{cases}$$

For $r = +\infty$, when $\alpha v_A \geq v_B$, $\beta\left(v_A, v_B, \alpha, r\right) = \frac{1}{2\alpha}\left(1 + \frac{v_B}{v_A}\right) v_B$; when $v_B \geq \alpha v_A$, $\beta\left(v_A, v_B, \alpha, r\right) = \frac{\alpha}{2}\left(1 + \frac{v_A}{v_B}\right) v_A$.

Property 1 says that the total effort exerted in each battle is proportional to the adjusted prize spread Δu. Given an arbitrary $r \in (0, +\infty)$, it follows from the characterization of $\beta\left(v_A, v_B, \alpha, r\right)$ that the sum of efforts in a one-shot contest is maximized when $\Delta u_B = \alpha^{1/r}\Delta u_A$, which implies that the optimal bias in battle t is $\alpha^* = \left(\Delta u_B/\Delta u_A\right)^r = \left(v_B/v_A\right)^r$ using the definition $\Delta u := \Delta u_A/v_A = \Delta u_B/v_B$. Analogously, given $r = +\infty$, the sum of efforts is maximized when $\alpha^* = v_B/v_A$. We summarize the result in the following.

Lemma 1. *In a single battle, the effort-maximizing bias is* $\alpha^*_{One\text{-}shot}(r) = \begin{cases} (v_B/v_A)^r, & \text{if } r \in (0,+\infty) \\ v_B/v_A, & \text{if } r = +\infty \end{cases}$.

To move on to a dynamic contest, we introduce the following notations. We use $V_A(n_1,n_2)$ (resp. $V_B(n_1,n_2)$) to denote the continuation value of team A (resp. team B) at state (n_1,n_2). At state (n_1,n_2), player $A(t)$'s effective prize spreads of winning the current battle is $\Delta u_A(n_1,n_2) = V_A(n_1+1,n_2) - V_A(n_1,n_2+1)$ and player $B(t)$'s effective prize spreads of winning this battle is $\Delta u_B(n_1,n_2) = V_B(n_1,n_2+1) - V_B(n_1+1,n_2)$. Given players' prize spreads $\Delta u_A(n_1,n_2)$ and $\Delta u_B(n_1,n_2)$, we compute their equilibrium efforts in this battle at state (n_1,n_2) using Property 1. Equipped with the notations, we analyze a best-of-three contest between two teams in the following.

Example 1. ***A Best-of-three Team Contest***

Consider a 3-battle team contest, i.e., $k=1$, each battle is modelled as a Tullock contest with $r \in (0,1]$.[23] If battles are homogeneous, i.e., α and r are the same across the state (n_1,n_2), the resulting total effort equals

$$TE_{Homo,k=1}(\alpha,r) = 6p_Ap_B\frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2},$$

where $p_A = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r}$ and $p_B = \frac{v_B^r}{\alpha v_A^r + v_B^r}$.

Proof. We solve the game backwards. For that, we compute players' effective prize spreads and their equilibrium efforts at each possible state $(1,1)$, $(1,0)$, $(0,1)$, $(0,0)$ as well as the probability of each state, respectively. The expected aggregate effort equals the weighted sum of the efforts at these states. Calculation details are relegated to subsection 5.2 in the appendix. □

More generally, when outcome-dependent heterogeneity is introduced, i.e., r can differ across the state (n_1,n_2) and/or α is allowed to be contingent on (n_1,n_2). For $r(n_1,n_2) \in (0,1]$, the resulting total effort equals

$$\begin{aligned} TE_{Hete,k=1}(\boldsymbol{\alpha},\mathbf{r}) \;=\; & [p_A(1,0)p_B(1,1) + p_B(0,1)p_A(1,1)]\,\beta(0,0) \\ & + p_A(0,0)p_B(1,1)\beta(1,0) + p_B(0,0)p_A(1,1)\beta(0,1) \end{aligned}$$

[23] In our general analysis, r can be any value within $(0,+\infty]$.

$$+[p_A(0,0)p_B(1,0)+p_B(0,0)p_A(0,1)]\beta\,(1,1)\,,$$

where $p_A(n_1,n_2)=\frac{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}}{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}+v_B^{r(n_1,n_2)}}$ and $\beta\,(n_1,n_2)=\frac{r\alpha(n_1,n_2)v_A^{r(n_1,n_2)}v_B^{r(n_1,n_2)}}{\left(\alpha v_A^{r(n_1,n_2)}+v_B^{r(n_1,n_2)}\right)^2}\,(v_A+v_B)$.

Proof. We characterize the subgame perfect equilibrium of this 3-battle contest by backward induction. Details are relegated to subsection 5.3 in the appendix. □

We next analyze a best-of-five team contest, i.e., $k=2$.

Example 2. ***A Best-of-five Team Contest***

Consider a team contest with 5 pairwise battles. The winning team is the team with at least three wins.

(i) With homogeneous battles, when each battle is modelled as a Tullock contest with $r\in(0,1]$, the expected aggregate effort equals

$$TE_{Homo,k=2}(\alpha,r)=30p_A^2p_B^2\frac{\alpha r v_A^r v_B^r\,(v_A+v_B)}{(\alpha v_A^r+v_B^r)^2},$$

where $p_A=\frac{\alpha v_A^r}{\alpha v_A^r+v_B^r}$ and $p_B=\frac{v_B^r}{\alpha v_A^r+v_B^r}$.

(ii) With outcome-dependent heterogeneity, when $r(n_1,n_2)$ can be any value within $(0,1]$, the expected aggregate effort equals

$$\begin{aligned}TE_{Hete,k=2}(\boldsymbol{\alpha},\mathbf{r}) &= TE(0,0)\\ &\quad +p_A(0,0)TE(1,0)+p_B(0,0)TE(0,1)\\ &\quad +P(2,0)TE(2,0)+P(0,2)TE(0,2)+P(1,1)TE(1,1)\\ &\quad +P(2,1)TE(2,1)+P(1,2)TE(1,2)\\ &\quad +P(2,2)TE(2,2),\end{aligned}$$

where

$$TE(0,0)=\left[\begin{array}{c}p_A(1,0)p_A(2,0)p_B(2,1)p_B(2,2)+p_A(1,0)p_B(1,1)p_A(2,1)p_B(2,2)\\ +p_A(1,0)p_B(1,1)p_B(1,2)p_A(2,2)+p_B(0,1)p_A(1,1)p_A(2,1)p_B(2,2)\\ +p_B(0,1)p_A(1,1)p_B(1,2)p_A(2,2)+p_B(0,1)p_B(0,2)p_A(1,2)p_A(2,2)\end{array}\right]\beta\,(0,0)\,;$$

$$
\begin{aligned}
TE(1,0) &= \left[\begin{array}{c} p_A(2,0)p_B(2,1)p_B(2,2)+p_B(1,1)p_A(2,1)p_B(2,2) \\ +p_B(1,1)p_B(1,2)p_A(2,2) \end{array}\right]\beta\left(1,0\right); \\
TE(0,1) &= \left[\begin{array}{c} p_A(1,1)p_A(2,1)p_B(2,2)+p_A(1,1)p_B(1,2)p_A(2,2) \\ +p_B(0,2)p_A(1,2)p_A(2,2) \end{array}\right]\beta\left(0,1\right); \\
TE(2,0) &= p_B(2,1)p_B(2,2)\beta\left(2,0\right); \\
TE(0,2) &= p_A(1,2)p_A(2,2)\beta\left(0,2\right); \\
TE(1,1) &= \left(p_A(2,1)p_B(2,2)+p_B(1,2)p_A(2,2)\right)\beta\left(1,1\right); \\
TE(2,1) &= p_B(2,2)\beta\left(2,1\right); \\
TE(1,2) &= p_A(2,2)\beta\left(1,2\right); \\
TE(2,2) &= \beta\left(2,2\right),
\end{aligned}
$$

and $P(n_1,n_2)$ is the probability of reaching the state (n_1,n_2) from $(0,0)$. For example, $P(2,0)=p_A(0,0)p_A(1,0)$, $P(0,2)=p_B(0,0)p_B(0,1)$, and $P(1,1)=p_A(0,0)p_B(1,0)+p_B(0,0)p_A(0,1)$.[24]

Proof. We characterize the subgame perfect equilibrium by backward induction. Details are relegated to subsections 5.4 and 5.5 in the appendix. □

When $k=1,\ 2$, we derive the expressions of $TE_{Homo,k}(\alpha,r)$ and $TE_{Hete,k}(\boldsymbol{\alpha},\mathbf{r})$ for $r\in(0,1]$ in Examples 1 and 2, respectively. Our examples reveal two difficulties in characterizing the optimal biases that maximize the expected aggregate effort for general cases. First, it is not an easy task to derive the explicit formulas of $TE_{Homo,k}(\alpha,r)$ and $TE_{Hete,k}(\boldsymbol{\alpha},\mathbf{r})$ for a general k and an arbitrarily given $r\in(0,+\infty]$. Solving for such a contest with $2k+1$ battles requires tracking players' incentives, computing their equilibrium efforts along each possible path, and determining the probability of each path. Consequently, the computation steps are inevitably long for a large k.[25] Second, given that the expression of the aggregate effort is sufficiently complicated, e.g., $TE_{Hete,k=2}(\boldsymbol{\alpha},\mathbf{r})$ in Example 2(ii), it is challenging to establish a general property of the optimal biases. Despite of the difficulties, we characterize the expected aggregate effort, identify the optimal bias for homogeneous case, and derive several

[24] We will formally introduce Definition 3 in Section 4.2 to compute $P(n_1,n_2)$ for general cases.
[25] See, for example, the proof of Example 2 in Section 5.5 in the appendix.

useful properties of the optimal outcome-dependent biases in the presence of heterogeneity in Section 4.

4 General Analysis for 2k+1 Battles

In this section, we consider a general dynamic contest between two teams with $2k+1$ pairwise battles, where k can be any positive integer. A team wins if and only if it wins at least $k+1$ out of $2k+1$ battles. To solve the expeced aggregate effort for such a contest, we compute players' equilibrium efforts at each possible state as well as the probability of each state. The solution concept is subgame perfect equilibrium. We will solve the contest game backwards.

To compute players' efforts, we track players' incentives. Specifically, we will compute their prize spreads at each possible state $(n_1, n_2) \in S$, where S is defined as an collection containing all possible states:

$$S := \{(n_1, n_2) | 0 \leq n_1, n_2 \leq k+1, \text{ and } n_1 + n_2 \leq 2k+1\}. \tag{2}$$

We further define the following set $E \subsetneq S$ as the collection of all ending nodes:

$$E := \{(n_1, n_2) | n_i = k+1 \text{ and } 0 \leq n_j \leq k-1\}.$$

A player's prize spread of winning battle $t = n_1 + n_2 + 1$ equals zero at an ending node, i.e., $\Delta u_A(n_1, n_2) = 0$ and $\Delta u_B(n_1, n_2) = 0$ at each $(n_1, n_2) \in E$. This is because $n_1 = k+1$ (resp. $n_2 = k+1$) means that team A (resp. team B) wins the entire contest by accumulating a majority number of battle wins.

At (k, k), it is straightforward to check $\Delta u_A(k, k) = v_A$ and $\Delta u_B(k, k) = v_B$. For other remaining states, we define G as the collection of all possible states excluding the ending nodes E and (k, k). Specifically,

$$G := S/(E \cup (k, k)) = \{(n_1, n_2) | n_1, n_2 \in Z, \, 0 \leq n_1, n_2 \leq k, \text{ and } n_1 + n_2 \leq 2k-1\}.$$

The collection of all possible states S can be partitioned into the collections E, $\{(k, k)\}$, and G, i.e., $S = E \cup \{(k, k)\} \cup G$. For each $(n_1, n_2) \in G$, we now provide the following recursive formula to compute the players' prize spreads.

Lemma 2. *For $(n_1, n_2) \in G$, players' effective prize spreads obey the following relation:*

$$\begin{aligned}\Delta u_A(n_1,n_2) &= p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1);\\ \Delta u_B(n_1,n_2) &= p_A(n_1+1,n_2)\Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1).\end{aligned} \quad (3)$$

Proof. Consider state (n_1, n_2) such that $n_1, n_2 \le k-1$, we prove the result in the following.

$$\begin{aligned}&\Delta u_A(n_1,n_2)\\ =\ & V_A(n_1+1,n_2) - V_A(n_1,n_2+1)\\ =\ & \begin{matrix} p_A(n_1+1,n_2)V_A(n_1+2,n_2) + p_B(n_1+1,n_2)V_A(n_1+1,n_2+1)\\ -p_A(n_1,n_2+1)V_A(n_1+1,n_2+1) - p_B(n_1,n_2+1)V_A(n_1,n_2+2)\end{matrix}\\ =\ & \begin{matrix} V_A(n_1+1,n_2+1) + p_A(n_1+1,n_2)\left(V_A(n_1+2,n_2) - V_A(n_1+1,n_2+1)\right)\\ -V_A(n_1,n_2+2) - p_A(n_1,n_2+1)\left(V_A(n_1+1,n_2+1) - V_A(n_1,n_2+2)\right)\end{matrix}\\ =\ & \Delta u_A(n_1,n_2+1) + p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) - p_A(n_1,n_2+1)\Delta u_A(n_1,n_2+1)\\ =\ & p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1).\end{aligned}$$

We complete the proof by proving the result for remaining cases. The details are relegated into the appendix. □

Lemma 3. *For any (n_1, n_2) such that $n_1, n_2 \le k$, we have*

$$\Delta u_A(n_1,n_2)/v_A = \Delta u_B(n_1,n_2)/v_B. \quad (4)$$

Proof. At (k,k), $\Delta u_A(k,k) = v_A$ and $\Delta u_B(k,k) = v_B$, and the formula holds automatically. We prove the lemma by mathematical induction. For (n_1, n_2) such that $n_1, n_2 \le k-1$, suppose that the lemma holds for (n_1+1, n_2) and (n_1, n_2+1), it suffices to show that the lemma holds for (n_1, n_2), which is true, since

$$\Delta u_A(n_1,n_2) = p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1);$$

$$\Delta u_B(n_1,n_2) \quad = \quad p_A(n_1+1,n_2)\Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1),$$

using Lemma 2. Details are provided in the appendix. □

By Lemma 3, two matched players always share a common *adjusted* prize spread across the states while its value may vary with the state. This result extends a key observation of Fu, Lu, and Pan (2015) into an asymmetric contest between two teams. Assuming that two teams value the final trophy equally, i.e., $v_A = v_B = 1$, Fu, Lu, and Pan (2015) show that two matched players always have a common value of winning a battle, regardless of the state. In contrast, we allow two teams to value the trophy differently, as a result, two matched players have different valuations of winning a battle. Nevertheless, we reveal that two matched players' valuations must obey (4), i.e., they always have a common *adjusted* prize spread, which suffices to capture players' incentives. We formally introduce the notation of the adjusted prize spread in the following.

Definition 1. *(Common Adjusted Prize Spread) Define the adjusted prize spread at a state* (n_1,n_2) *as*

$$\Delta u(n_1,n_2) := \Delta u_A(n_1,n_2)/v_A = \Delta u_B(n_1,n_2)/v_B. \tag{5}$$

From (5) and Property 1, one can easily solve for players' equilibrium efforts at an arbitrary state (n_1,n_2), given the adjusted prize spread $\Delta u(n_1,n_2)$. Up to now, we have discussed how to simplify the way to solve players' equilibrium effort in a battle at each possible state $(n_1,n_2) \in S$. To derive the expected aggregate effort, it remains to derive the winning probability of each battle at each possible state. We therefore present the following result.

Lemma 4. *For any* (n_1,n_2) *such that* $n_1, n_2 \leq k$*, player* $A(t)$*'s winning probability of a* *battle is solely determined by* $v_A, v_B, \alpha(n_1,n_2), r(n_1,n_2)$*. More precisely,*

$$p_A(n_1,n_2) = \begin{cases} \frac{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}}{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}+v_B^{r(n_1,n_2)}}, & \textit{if } r \in (0, \widehat{r}(\frac{v_B}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A})] \\ & \textit{and } [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A \geq v_B; \\ \frac{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}}{\alpha(n_1,n_2)v_A^{r(n_1,n_2)}+v_B^{r(n_1,n_2)}}, & \textit{if } r \in (0, \widehat{r}(\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}{v_B})] \\ & \textit{and } v_B \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A; \\ 1-\left(1-\frac{1}{r}\right)\left(\frac{1}{r-1}\right)^{1/r}\frac{v_B}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}, & \textit{if } r \in (\widehat{r}(\frac{v_B}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}), 2] \\ & \textit{and } [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A \geq v_B; \\ \left(1-\frac{1}{r}\right)\left(\frac{1}{r-1}\right)^{1/r}\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}{v_B}, & \textit{if } r \in (\widehat{r}(\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}{v_B}), 2] \\ & \textit{and } v_B \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A; \\ 1-\frac{1}{2}\frac{v_B}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}, & \textit{if } r \in (2,+\infty) \\ & \textit{and } [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A \geq v_B; \\ \frac{1}{2}\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}v_A}{v_B}, & \textit{if } r \in (2,+\infty) \\ & \textit{and } v_B \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\, v_A. \\ 1-\frac{1}{2}\frac{v_B}{\alpha(n_1,n_2)v_A}, & \textit{if } r = +\infty \textit{ and } \alpha(n_1,n_2)v_A \geq v_B; \\ \frac{1}{2}\frac{\alpha(n_1,n_2)v_A}{v_B}, & \textit{if } r = +\infty \textit{ and } v_B \geq \alpha(n_1,n_2)v_A. \end{cases}$$

At state (n_1,n_2), player $A(t)$'s probability of winning this battle t is $p_A(n_1,n_2)$, which equals

$$\begin{cases} \frac{\alpha(n_1,n_2)[\Delta u_A(n_1,n_2)]^{r(n_1,n_2)}}{\alpha(n_1,n_2)[\Delta u_A(n_1,n_2)]^{r(n_1,n_2)}+[\Delta u_B(n_1,n_2)]^{r(n_1,n_2)}}, & \text{if } r \in (0, \widehat{r}(\frac{\Delta u_B(n_1,n_2)}{\alpha(n_1,n_2)\Delta u_A(n_1,n_2)})] \text{ and } \\ & [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2) \geq \Delta u_B(n_1,n_2); \\ \frac{\alpha(n_1,n_2)[\Delta u_A(n_1,n_2)]^{r(n_1,n_2)}}{\alpha(n_1,n_2)[\Delta u_A(n_1,n_2)]^{r(n_1,n_2)}+[\Delta u_B(n_1,n_2)]^{r(n_1,n_2)}}, & \text{if } r \in (0, \widehat{r}(\frac{\alpha(n_1,n_2)\Delta u_A(n_1,n_2)}{\Delta u_B(n_1,n_2)})] \text{ and } \\ & \Delta u_B(n_1,n_2) \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2); \\ 1-\left(1-\frac{1}{r}\right)\left(\frac{1}{r-1}\right)^{1/r}\frac{\Delta u_B(n_1,n_2)}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}, & \text{if } r \in (\widehat{r}(\frac{\Delta u_B(n_1,n_2)}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}), 2] \text{ and } \\ & [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2) \geq \Delta u_B(n_1,n_2); \\ \left(1-\frac{1}{r}\right)\left(\frac{1}{r-1}\right)^{1/r}\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}{\Delta u_B(n_1,n_2)}, & \text{if } r \in (\widehat{r}(\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}{\Delta u_B(n_1,n_2)}), 2] \text{ and } \\ & \Delta u_B(n_1,n_2) \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2); \\ 1-\frac{1}{2}\frac{\Delta u_B(n_1,n_2)}{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}, & \text{if } r \in (2,+\infty) \text{ and } \\ & [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2) \geq \Delta u_B(n_1,n_2); \\ \frac{1}{2}\frac{[\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2)}{\Delta u_B(n_1,n_2)}, & \text{if } r \in (2,+\infty) \text{ and } \\ & \Delta u_B(n_1,n_2) \geq [\alpha(n_1,n_2)]^{1/r(n_1,n_2)}\Delta u_A(n_1,n_2). \\ 1-\frac{1}{2}\frac{\Delta u_B(n_1,n_2)}{\alpha(n_1,n_2)\Delta u_A(n_1,n_2)}, & \text{if } r = +\infty \text{ and } \\ & \alpha(n_1,n_2)\Delta u_A(n_1,n_2) \geq \Delta u_B(n_1,n_2); \\ \frac{1}{2}\frac{\alpha(n_1,n_2)\Delta u_A(n_1,n_2)}{\Delta u_B(n_1,n_2)}, & \text{if } r = +\infty \text{ and } \\ & \Delta u_B(n_1,n_2) \geq \alpha(n_1,n_2)\Delta u_A(n_1,n_2). \end{cases}$$

(5) implies that $\Delta u_i(n_1, n_2) = \Delta u(n_1, n_2)v_i$ for $i \in \{A, B\}$. The lemma thus follows.

Lemma 4 shows that how the equilibrium winning likelihoods $(p_A(n_1, n_2), p_B(n_1, n_2))$ depend on the state-specific characteristics, $\alpha(n_1, n_2)$ and $r(n_1, n_2)$, where $p_B(n_1, n_2) = 1 - p_A(n_1, n_2)$, given the ratio of v_A/v_B. As a result, the outcome-dependent heterogeneity would directly lead to the failure of the history independence result in team contests.[26]

By (3) and (5), we rewrite the recursive formula in Lemma 2 as

$$\Delta u(n_1, n_2) = p_A(n_1 + 1, n_2)\Delta u(n_1 + 1, n_2) + p_B(n_1, n_2 + 1)\Delta u(n_1, n_2 + 1), \quad (6)$$

where $p_A(n_1 + 1, n_2)$ and $p_B(n_1, n_2 + 1) = 1 - p_A(n_1, n_2 + 1)$ are given by Lemma 4. We will rely on the formula (6) to derive the expected total effort in the following analysis.

[26] Fu, Lu, and Pan (2015) eastablish the history independence result in an unbiased team contest. Note that the result of history independence holds for a biased team contest with homogeneous battles, but collapses in the presence of outcome-dependent heterogeneity.

4.1 Homogeneous Case

If α and r are the same across the state (n_1, n_2), $p := p_A(n_1, n_2)$ must remain the same across the state using Lemma 4, and therefore the expected total effort can be simplified as

$$\mathbf{TE}_{Homo,k}(\alpha, r) = \sum_{(n_1,n_2)\in S/E} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} E(n_1, n_2), \tag{7}$$

where $E(n_1, n_2)$ is the expected effort from the paired players in a battle at state (n_1, n_2).

Proposition 1. *The total effort function in (7) equals*

$$\mathbf{TE}_{Homo,k}(\alpha, r) = (2k+1) C_{2k}^{k} p^{k} (1-p)^{k} \beta(v_A, v_B, \alpha, r), \tag{8}$$

where the total number of battles is 2k+1 and $\beta(v_A, v_B, \alpha, r)$ is given by Property 1.

Proof. By (5) and Property 1, we first rewrite $\mathbf{TE}_{Homo,k}(\alpha, r)$ in (7) as

$$\begin{aligned}
\mathbf{TE}_{\text{Homo}} &= \sum_{(n_1,n_2)\in S/E} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} E(n_1, n_2) \\
&= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} \beta(v_A, v_B, \alpha, r) \Delta u(n_1, n_2).
\end{aligned}$$

To simplify the calculation, we show two useful results in the following two steps. In Step 1, we prove $\Delta u(n_1, n_2) = \gamma(n_1, n_2) p^{k-n_1} (1-p)^{k-n_2}$. In Step 2, we show that $C_{2k}^{k} = \sum_{n_1+n_2=t} C_{n_1+n_2}^{n_1} \gamma(n_1, n_2)$ holds for any $t \in \{1, 2, ..., 2k\}$. Combining the results, we have

$$\begin{aligned}
\mathbf{TE}_{\text{Homo}} &= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} \beta(v_A, v_B, \alpha, r) \Delta u(n_1, n_2) \\
&= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} \gamma(n_1, n_2) \beta(v_A, v_B, \alpha, r) p^{k} (1-p)^{k} \\
&= \sum_{t=1}^{2k} \sum_{n_1+n_2=t} C_{n_1+n_2}^{n_1} \gamma(n_1, n_2) \beta(v_A, v_B, \alpha, r) p^{k} (1-p)^{k} \\
&= \sum_{t=1}^{2k} C_{2k}^{k} \beta(v_A, v_B, \alpha, r) p^{k} (1-p)^{k}
\end{aligned}$$

$$= \quad (2k+1)C_{2k}^{k}p^{k}\left(1-p\right)^{k}\beta(v_A, v_B, \alpha, r).$$

The details are relegated into the appendix. □

We now formally describe the problem as follows. With homogeneous battles, given $r \in (0, +\infty]$, the optimal bias α solves the following program:

$$\mathbb{P}_{Homo} := \max_{\alpha \in [0,+\infty)} \mathbf{TE}_{Homo,k}(\alpha, r). \tag{9}$$

Theorem 1. *With homogeneous battles, the optimal bias that maximizes the expected aggregate effort $\mathbf{TE}_{Homo,k}(\alpha, r)$ in (7) is $\alpha^* = \alpha^*_{One\text{-}shot}(r)$.*

Proof. The optimal bias α^* solves the program in (9). Recall in Proposition 1, $\mathbf{TE}_{Homo,k}(\alpha, r) = (2k+1)C_{2k}^{k}p^{k}\left(1-p\right)^{k}\beta$. It then suffices to maximize $p^{k}\left(1-p\right)^{k}\beta(v_A, v_B, \alpha, r)$ by choosing a $\alpha \in [0, +\infty)$. On one hand, $\alpha^*_{\text{One-shot}}(r)$ maximizes $\beta(v_A, v_B, \alpha, r)$ for any given $r \in (0, +\infty]$. This is because β is maximized when $\Delta u_B = \alpha^{1/r}\Delta u_A$, i.e., $\alpha = (v_B/v_A)^r$ for $r \in (0, +\infty)$ and when $\Delta u_B = \alpha \Delta u_A$, i.e., $\alpha = v_B/v_A$ for $r = +\infty$. On the other hand, $p^{k}\left(1-p\right)^{k}$ or $p\left(1-p\right)$ is maximized when $p = 1/2$, which holds true when $\alpha = \alpha^*_{\text{One-shot}}(r)$. Therefore, $\alpha^*_{\text{One-shot}}(r)$ is the optimal bias. □

Compare to Lemma 1, we conclude that there is no loss to search for the effort-maximizing bias of a single battle when all battles are homogeneous. Nevertheless, the intuition can be different: From the proof of Theorem 1, the bias $\alpha^*_{\text{One-shot}}$ maximizes not only the effort in each battle, but also $C_{2k}^{k}p^{k}\left(1-p\right)^{k}$, the probability of reaching the state (k, k), at which competition is the fiercest. Recall that the contest quits whenever a team accumulates $k+1$ wins. Certeris paribus, the more battles are played, the greater effort the designer could elicit.

4.2 Heterogeneous Case

For the sake of generality, we allow outcome-dependent heterogeneity in this subsection, which means that r can vary with the state (n_1, n_2) and/or α is allowed to be contingent on (n_1, n_2). Due to the failure of history independence result, we have to trace the history of battle outcomes in this case, as players' efforts depend on the path of history. For that, we define *path* g as a sequence of the *ordered* nodes, i.e., $g = \{(n_1^1, n_2^1), ..., (n_1^t, n_2^t), ..., (n_1^T, n_2^T)\}$, where $(n_1^t, n_2^t) \in S$, $\forall t \in \{1, ..., T\}$. Recall that S defined in (2) is an collection containing all possible states. On this basis, we introduce the definition of a feasible path in the following.

Definition 2. *(Feasible Path) A path* $g = \{(n_1^1, n_2^1), ..., (n_1^t, n_2^t), (n_1^{t+1}, n_2^{t+1}), ..., (n_1^T, n_2^T)\}$ *is feasible if* (n_1^{t+1}, n_2^{t+1}) *equals either* (n_1^t+1, n_2^t) *or* (n_1^t, n_2^t+1), $\forall t \in \{1, ..., T\}$, *where* $T \leq k+1$. *Denote* $G((s_1, s_2), (e_1, e_2))$ *as a collection of all feasible paths starting from* (s_1, s_2) *to* (e_1, e_2).

The feasiblity constraints basically require an uninterrupted description of the battle history: each state (n_1^t, n_2^t) reaches either $(n_1^t + 1, n_2^t)$ or $(n_1^t, n_2^t + 1)$ after battle t is played, since either team A or team B wins the battle. Note that (n_1^1, n_2^1) is not necessarily $(0, 0)$ and (n_1^T, n_2^T) is not necessarily an ending state of the whole contest. For example, $\{(1, 0)$, $(1, 1)$, $(1, 2)\}$ describes a feasible path in a 3-battle contest (i.e., $k = 1$), $\{(2, 0)$, $(2, 1)$, $(2, 2)\}$ describes a feasible path in a 5-battle contest (i.e., $k = 2$), and $G((0, 0), (1, 1)) = \{g_1, g_2\}$, where $g_1 = \{(0, 0), (1, 0), (1, 1)\}$ and $g_2 = \{(0, 0), (0, 1), (1, 1)\}$.

For an easier exposition, we further define the probability and the adjusted probability of a feasible path in the following, which will play a crucial role in our following analysis.

Definition 3. *(Probability of A Path) (i) Given a feasible path* $g = \{(n_1^1, n_2^1),...,(n_1^t, n_2^t),...,(n_1^T, n_2^T)\}$, *denote* $P(g) := \Pi_{t=1}^{T-1} p((n_1^t, n_2^t), (n_1^{t+1}, n_2^{t+1}))$ *as the probability of path* g, *where*

$$p((n_1^t, n_2^t), (n_1^{t+1}, n_2^{t+1})) = \begin{cases} p_A(n_1^t, n_2^t), & \text{if } (n_1^{t+1}, n_2^{t+1}) = (n_1^t + 1, n_2^t); \\ p_B(n_1^t, n_2^t), & \text{if } (n_1^{t+1}, n_2^{t+1}) = (n_1^t, n_2^t + 1). \end{cases}$$

Denote $\widehat{P}(g) := \Pi_{t=1}^{T-1} \widehat{p}((n_1^t, n_2^t), (n_1^{t+1}, n_2^{t+1}))$ *as the adjusted probability of path* g, *where*

$$\widehat{p}((n_1^t, n_2^t), (n_1^{t+1}, n_2^{t+1})) = \begin{cases} p_A(n_1^t + 1, n_2^t), & \text{if } (n_1^{t+1}, n_2^{t+1}) = (n_1^t + 1, n_2^t); \\ p_B(n_1^t, n_2^t + 1), & \text{if } (n_1^{t+1}, n_2^{t+1}) = (n_1^t, n_2^t + 1). \end{cases}$$

(ii) For any reachable state (n_1^t, n_2^t), $P((n_1^t, n_2^t), (n_1^t, n_2^t)) := 1$ *and* $\widehat{P}((n_1^t, n_2^t), (n_1^t, n_2^t)) := 1$.

(iii) For any infeasible path g, $P(g) := 0$ *and* $\widehat{P}(g) := 0$.

Consider the following examples. By Definition 3(i), given $g_1 = \{(0, 0), (1, 0), (1, 1)\}$, the probability of path g_1 is $P(g_1) = p_A(0, 0)p_B(1, 0)$ and the adjusted probability of path g_1 is $\widehat{P}(g_1) = \widehat{p}((0, 0), (1, 0))\widehat{p}((1, 0), (1, 1)) = p_A(1, 0)p_B(1, 1)$. Given $g_2 = \{(0, 0), (0, 1), (1, 1)\}$, the probability of path g_2 is $P(g_2) = p_B(0, 0)p_A(0, 1)$ and the adjusted probability of path g_2 is $\widehat{P}(g_2) = \widehat{p}((0, 0), (0, 1))\widehat{p}((0, 1), (1, 1)) = p_B(0, 1)p_A(1, 1)$.

By Definition 3(ii), $P((0, 0), (0, 0)) = 1$ and $\widehat{P}((k, k), (k, k)) = 1$. By Definition 3(iii), $P((1, 0), (0, 1)) = 0$ and $\widehat{P}((1, 0), (0, 1)) = 0$, since the path $\{(1, 0), (0, 1)\}$ is infeasible.

Based on the definitions, we will propose a general program whose solutions give the optimal vector of biases in the next subsection.

4.2.1 General Program

Equipped with the notations in Definitions 2 and 3, we first characterize the expected aggregate effort in the following proposition.

Proposition 2. *The expected aggregate effort equals*

$$\mathbf{TE}_{Hete,k} = \sum_{(n_1,n_2)\in S/E} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2), \tag{10}$$

where $P(g)$ and $\widehat{P}(g)$ are defined in Defnition 3.

Proof. We first reformulate the expected aggregate effort as

$$\mathbf{TE}_{Hete,k} = \sum_{(n_1,n_2)\in S/E} P(n_1,n_2)E(n_1,n_2),$$

where S/E is the collection of all possible states excluding the ending ones, $P(n_1,n_2)$ is the probability that the contest reaches state (n_1,n_2), and $E(n_1,n_2)$ is the expected sum of efforts from the two matched players in a battle t at state (n_1,n_2). In particular, it follows from direct calculation that $P(n_1,n_2) = \sum_{g\in G((0,0),(n_1,n_2))} P(g)$ and $E(n_1,n_2) = \beta(n_1,n_2)\Delta u(n_1,n_2)$ using Property 1. To establish Proposition 2, it only remains to show that

$$\Delta u(n_1,n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g). \tag{11}$$

We relegate the full proof into the appendix. □

Proposition 2 reveals how the choice of biases $\{\alpha(n_1,n_2)\}_{(n_1,n_2)\in S/E}$ affects $\mathbf{TE}_{Hete,k}$. To be more specific, the biases impact the expected aggregate effort through two channels: by influencing the likelihood of reaching a state and by affecting players' incentive for exerting efforts at that state, i.e., $P(n_1,n_2)$ and $\Delta u(n_1,n_2)$. In the proof of Proposition 2, we show that $\Delta u(n_1,n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$, which implies that the players' incentives for exerting effort in the current battle depends on the winning probabilities at attainable states in subsequent battles.

When α and r do not depend on the state (n_1, n_2), both the winning probability $p := p_A(n_1, n_2)$ and $\beta := \beta(n_1, n_2)$ are invariant across the states. In this case, we show that the aggregate effort function in (10) boils down to (7) for homogeneous battles. The result is summarized in the following corollary.

Corollary 1. *When α and r do not depend on the state (n_1, n_2), the expected aggregate effort function in (10) coincides with that in (7).*

Proof. See the appendix. □

The Corollary above shows that when both α and r are invariant of the state (n_1, n_2), our general program coincides to the one for homogeneous battle in Corollary 1. When α and/or r can vary with the state (n_1, n_2), we introduce the following program to search the optimal biases.

With Proposition 2, we formally formulate the program for team contests with outcome-dependent heterogeneity as follows. Given $\mathbf{r} = \{r(n_1, n_2)\}_{(n_1,n_2)\in S/E}$ with $r(n_1, n_2) \in (0, +\infty]$, the optimal biased rule $\boldsymbol{\alpha} = \{\alpha(n_1, n_2)\}_{(n_1,n_2)\in S/E}$ solves the following program:

$$\begin{aligned} \mathbb{P}_{Hete} \quad : \quad &= \max_{\boldsymbol{\alpha}} \mathbf{TE}_{Hete,k}(\boldsymbol{\alpha}, \mathbf{r}) \\ s.t. \qquad & \alpha(n_1, n_2) \in [0, +\infty), \text{ where } (n_1, n_2) \in S/E. \end{aligned} \tag{12}$$

We next show the existence of the optimal biases that solve the program described in (12) as follows.

Proposition 3. *There always exists a optimal biased rule $\boldsymbol{\alpha}^* = \{\alpha^*(n_1, n_2)\colon (n_1, n_2) \in S/E\}$ that solves the program $\mathbb{P}_{Hete}$ in (12).*

Proof. Since $p_A(n_1, n_2)$ in Lemma 4 and $\beta\left(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2)\right)$ are both continuous in $\alpha(n_1, n_2)$, $\mathbf{TE}_{Hete,k}(\boldsymbol{\alpha}, \mathbf{r})$ in (10) is continuous in $\boldsymbol{\alpha}$ for any fix $\mathbf{r}$. In addition, $\lim_{\boldsymbol{\alpha}\to+\infty} \mathbf{TE}_{Hete,k}(\boldsymbol{\alpha}, \mathbf{r}) = 0$, as $|\mathbf{TE}_{Hete,k}(\boldsymbol{\alpha}, \mathbf{r})| \le \sum_{(n_1,n_2)\in S/E} \beta(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2))$ and $\lim_{\alpha(n_1,n_2)\to+\infty} \beta(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2)) = 0$. This implies that there must exist a $M > 0$ such that $\mathbf{TE}_{Hete,k}(\boldsymbol{\alpha}, \mathbf{r}) < \mathbf{TE}_{Hete,k}(\mathbf{M}, \mathbf{r})$. By Weierstrass Extreme Value Theorem, there must exist an optimal $\boldsymbol{\alpha} \in [0, M]^{|S/E|}$ that maximizes the total expected effort in (10). □

So far we have introduced the general program (12) whose solutions yield the optimal biases and establish the existence of the optimum in Proposition 3. However, we are encountering challenges in characterizing the optimal biased rule for two primary reasons. First, the general program outlined in (12) we are using to solve the optimal biases is multidimensional and computationally complex. Second, it remains unclear whether we can identify a valuable property that will help in comprehending how outcome-dependent heterogeneity impacts the design of biased rules. Even for $k = 2$, it is not straightforward to solve for the optimal biases, since the formula of $\mathbf{TE}_{Hete,k=2}(\boldsymbol{\alpha}, \mathbf{r})$ is sufficiently complicated (See Example 2). To tackle the problems, we will simplify the general program and explore the general properties of the optimal biases in the next subsection.

4.2.2 Simplification and Properties of Optimal Biases

In this subsection, we will decompose the multi-dimensional program in (12) into a system of single-dimensional optimization subproblems in Proposition 4. This simplification offers a way to search for the optimal biases. Moreover, we establish the two general properties of the optimums in Corollary 3 and Corollary 4.

To proceed, we will first identify the effect of a particular bias $\alpha(n_1', n_2')$ on the expected aggregate effort, which plays an important role in simplifying our general problem. Recall that Proposition 2 reveals how biased rule $\boldsymbol{\alpha} = \{\alpha(n_1, n_2)\}_{(n_1,n_2)\in S/E}$ affects the expected aggregate effort $\mathbf{TE}_{Hete,k}$. However, we are still uncertain about how a change in a particular bias $\alpha(n_1', n_2')$ influences the expected aggregate effort. We first note that the bias $\alpha(n_1', n_2')$ affects $\mathbf{TE}_{Hete,k}$ in (10) through $p_A(n_1', n_2')$ and $\beta(n_1', n_2')$.[27] The effect on $\beta(n_1', n_2')$ is measured by the coefficient $\omega(\boldsymbol{\alpha}/\alpha(n_1', n_2')) := \Big(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\Big)\Big(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g)\Big)$, where $\boldsymbol{\alpha}/\alpha(n_1', n_2')$ denotes the vector $\{\alpha(n_1, n_2)\}_{(n_1,n_2)\in S/(E\cup(n_1',n_2'))}$. It then remains to analyze how $p_A(n_1', n_2')$ enters in $\mathbf{TE}_{Hete,k}$, which is solved by the following proposition.

Lemma 5. *The coefficient of $p_A(n_1', n_2')$ in $\mathbf{TE}_{Hete,k}$ in (10) is $\varphi(\boldsymbol{\alpha}/\alpha(n_1', n_2')) = I + II$, where*

$$I = \sum_{t\in\{t'+1,\dots,2k\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left[\begin{array}{c} \Big(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\Big) \left(\begin{array}{c} \sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) \\ -\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \end{array} \right) \\ \cdot\Delta u(n_1,n_2)\beta(n_1,n_2) \end{array} \right],$$

[27]Note that the prize spread $\Delta u(n_1, n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ does not depend on the bias $\alpha(n_1', n_2')$.

and

$$II = \sum_{t\in\{0,\ldots,t'-1\}} \sum_{\substack{n_1+n_2=t\\(n_1,n_2)\in S/E}} \left[\begin{array}{c} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right) \left(\begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) \\ -\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) \end{array} \right) \\ \cdot\Delta u(n_1',n_2')\beta(n_1,n_2) \end{array} \right].$$

Recall that $\Delta u(n_1,n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ *is defined in (11).*

Proof. The total effort function in (10) can be rewritten as

$$\mathbf{TE}_{Hete,k} = \sum_{(n_1,n_2)\in S/E} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2).$$

Depending on the state (n_1,n_2), $p_A(n_1',n_2')$ may affect the probability of the paths and players' incentive along the paths, i.e., $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ and $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$. Recall that, by Definition 3, $p((n_1',n_2'),(n_1'+1,n_2')) = p_A(n_1',n_2')$, $p((n_1',n_2'),(n_1',n_2'+1)) = p_B(n_1',n_2')$, $\widehat{p}((n_1'-1,n_2'),(n_1',n_2')) = p_A(n_1',n_2')$, and $\widehat{p}((n_1',n_2'-1),(n_1',n_2')) = p_B(n_1',n_2')$.

We show that $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ is a linear function of $p_A(n_1',n_2')$ for any (n_1,n_2) and the coefficient of $p_A(n_1',n_2')$ in $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ equals

$$\left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g) \right) \left(\sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) - \sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \right).$$

Moreover, $p_A(n_1',n_2')$ would only affect the probability of a path after battle $t' = n_1'+n_2'+1$, the effect of $p_A(n_1',n_2')$ on the expected aggregate effort through affecting the probability of paths can be summarized by

$$I = \sum_{t\in\{t'+1,\ldots,2k\}} \sum_{\substack{n_1+n_2=t\\(n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g) \right) \left(\begin{array}{c} \sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) \\ -\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \end{array} \right)$$

$$\cdot\Delta u(n_1,n_2)\beta(n_1,n_2),$$

using $\Delta u(n_1,n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$.

Analogously, we show that $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ is a linear function of $p_A(n_1',n_2')$ for any

(n_1, n_2) and the coefficient of $p_A(n_1', n_2')$ in $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ equals

$$\left(-\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) + \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g)\right)\left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g)\right).$$

Moreover, $p_A(n_1', n_2')$ would only affect the incentives, i.e., $\widehat{P}(g)$ before battle $t' - 1 = n_1' + n_2'$, the effect of $p_A(n_1', n_2')$ on the expected aggregate effort through affecting the players' incentive along the paths can be summarized by

$$\begin{aligned} II \quad = \quad & \sum_{t\in\{0,\ldots,t'-1\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right) \\ & \cdot \left(\begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) \\ -\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) \end{array}\right) \Delta u(n_1', n_2')\beta(n_1, n_2). \end{aligned}$$

Combining the results above, we conclude that the coefficient of $p_A(n_1', n_2')$ in $\mathbf{TE}_{Hete,k}$ is $I + II$. More details are relegated into the appendix. □

By Lemma 5, we qualify the effect of $p_A(n_1, n_2)$ on the expected aggregate effort $\mathbf{TE}_{Hete,k}$, which is captured by $\varphi(\boldsymbol{\alpha}/\alpha(n_1, n_2))$. Certeris paribus, if $p_A(n_1, n_2)$ changes into $p_A(n_1, n_2)+\Delta$, the aggregate effort increases from $\mathbf{TE}_{Hete,k}$ to $\mathbf{TE}_{Hete,k}+\varphi(\boldsymbol{\alpha}/\alpha(n_1, n_2))\Delta$. Furthermore, the effect $\varphi(\boldsymbol{\alpha}/\alpha(n_1, n_2))$ can be decomposed into I and II. In particular, I measures the effect of a change in $p_A(n_1, n_2)$ on the probability of a path and II measures the effect on the players' incentives, since $\Delta u(n_1, n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ depends on the probability of the paths involved.

More specifically, Lemma 5 implies that a change in $p_A(n_1, n_2)$ affects the aggregate effort in (10) through two channels. First, a change in $p_A(n_1, n_2)$ has an impact on the probability of reaching a state in subsequent battles. Consider a 3-battle contest, as $p_A(1, 0)$ increases, the contest is more likely to reach the state $(2, 0)$, rather than $(1, 1)$. Note that players would spend more efforts on $(1, 1)$ than on $(2, 0)$. Second, a change in $p_A(n_1, n_2)$ presumably influences players' incentives for exerting effort at a state *before* battle $t = n_1 + n_2 + 1$. For example, in a 3-battle contest, a high $p_A(1, 1)$ would lower players' incentive for exerting effort at state $(1, 0)$. This is because even player $B(2)$ wins battle 2, team B is less likely to win battle 3 with a higher $p_A(1, 1)$, which in turn lowers player $B(2)$'s incentive and therefore that of player $A(2)$ for exerting effort. This intuition can be further confirmed by $\Delta u(1,0) = p_B(1,1)$.

Lemma 5 yields an important implication that $\mathbf{TE}_{Hete,k}$ in (10) can be decomposed into $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2)$ and other terms that do not involve $\alpha(n_1,n_2)$. In other words, $\alpha(n_1,n_2)$ influences the aggregate effort by affecting $p_A(n_1,n_2)$ and $\beta(n_1,n_2)$. More detailly, as $\alpha(n_1,n_2)$ increases by Δ, the change in the aggregate effort equals $[\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))\partial p_A(n_1,n_2)/\partial\alpha(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\partial\beta(n_1,n_2)/\partial\alpha(n_1,n_2)]\,\Delta$ approximately, where $p_A(n_1,n_2)$ is given by lemma 4 and $\beta(n_1,n_2)=\beta\left(v_A,v_B,\alpha(n_1,n_2),r(n_1,n_2)\right)$. By applying the aforementioned decomposition, we simplify the multi-dimensional program described in (12) into a system of single-dimensional optimization subproblems in the following proposition.

Proposition 4. *The $|S/E|$-dimensional problem in (12) can be converted into a system of $|S/E|$ single-dimensional optimization subproblems, where each subprogram is given by*

$$\max_{\alpha(n_1,n_2)\in[0,+\infty)} \varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2). \tag{13}$$

where $p_A(n_1,n_2)$ is given by lemma 4, $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))$ is given by Lemma 5 and $\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2)):=\left(\sum_{g\in G((0,0),(n_1,n_2))}P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))}\widehat{P}(g)\right)$.

Proof. By Lemma 5, $\mathbf{TE}_{Hete,k}$ in (10) can be decomposed into $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2)$ and other terms that do not involve $\alpha(n_1,n_2)$. In particular, neither $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))$ nor $\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))$ depends on $\alpha(n_1,n_2)$, where $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))=I+II$ and $\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2)):=\left(\sum_{g\in G((0,0),(n_1,n_2))}P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))}\widehat{P}(g)\right)$. Therefore, it suffices to consider $\max_{\alpha(n_1,n_2)\in[0,+\infty)}\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2), \forall(n_1,n_2)\in S/E$. □

It is worth noting that the solution to the general problem (12) must solve the system of these subproblems described in (13).[28] We can therefore rely on Proposition 4 to search for and to learn more about the optimal biases in such a team contest with outcome-dependent heterogeneity. If $\beta(n_1,n_2):=\beta\left(v_A,v_B,\alpha(n_1,n_2),r(n_1,n_2)\right)$ is differentiable, we can further apply first-order conditions to simplify the single-dimensional optimization subproblems into a system of equations in the following.

[28] If the solution to the system of these subproblems in (13) is unique, it must be the optimal biased rule that solves (12).

Corollary 2. *When $r(n_1, n_2) \leq 1$, the optimal biased rule must satisfy the system of $|S/E|$ equations:*

$$\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))\frac{v_A^r v_B^r}{\left(v_B^r + \alpha(n_1,n_2)v_A^r\right)^2} + \omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\frac{d\beta\left(n_1,n_2\right)}{d\alpha(n_1,n_2)} = 0, \text{ where } (n_1,n_2) \in S/E,$$

where $\beta(n_1, n_2) := \beta\left(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2)\right)$ is differentiable with respect to $\alpha(n_1, n_2)$, when $r(n_1, n_2) \leq 1$.

Corollary 2 provides a practical way to search for the optimal biases. Equipped with Proposition 4 and Proposition 3, we establish two general properties in Corollary 3 and Corollary 4, which shed light on the optimal design of the biases.

Corollary 3. *When r takes different values over (n_1, n_2), i.e., $r(n_1, n_2) \neq r(n_1', n_2')$ whenever $(n_1, n_2) \neq (n_1', n_2')$, the biased rule $\boldsymbol{\alpha}^*_{Homo} = \{\alpha(n_1, n_2) | \alpha(n_1, n_2) = \alpha^*_{One\text{-}shot}(r(n_1, n_2))\}$ would never be the optimal.*

Proof. We plug $\boldsymbol{\alpha}^*_{Homo}$ into $\varphi(\boldsymbol{\alpha}/\alpha(n_1, n_2))$ given by Lemma 5, we have $\varphi(\boldsymbol{\alpha}^*_{Homo}/\alpha(n_1, n_2)) = I + II$, where

$$\begin{aligned} I &= \left(\frac{1}{2}\right)^{n_1+n_2}\left[\begin{array}{c} \Delta u(n_1+1,n_2)\beta(n_1+1,n_2)\mathbf{1}\{(n_1+1,n_2)\in S/E\} \\ -\Delta u(n_1,n_2+1)\beta(n_1,n_2+1)\mathbf{1}\{(n_1,n_2+1)\in S/E\} \end{array}\right] \\ &= \left(\frac{1}{2}\right)^{2k-1}\left[\beta(n_1+1,n_2)\mathbf{1}\{(n_1+1,n_2)\in S/E\} - \beta(n_1,n_2+1)\mathbf{1}\{(n_1,n_2+1)\in S/E\}\right], \end{aligned}$$

and

$$\begin{aligned} II &= \left(\frac{1}{2}\right)^{n_1+n_2}\left[\begin{array}{c} \Delta u(n_1-1,n_2)\beta(n_1-1,n_2)\mathbf{1}\{(n_1-1,n_2)\in S/E\} \\ -\Delta u(n_1,n_2-1)\beta(n_1,n_2-1)\mathbf{1}\{(n_1,n_2-1)\in S/E\} \end{array}\right] \\ &= \left(\frac{1}{2}\right)^{2k+1}\left[\beta(n_1-1,n_2)\mathbf{1}\{(n_1-1,n_2)\in S/E\} - \beta(n_1,n_2-1)\mathbf{1}\{(n_1,n_2-1)\in S/E\}\right], \end{aligned}$$

where $\mathbf{1}\{\cdot\}$ is an indicator function. As a result,

$$\begin{aligned} &\varphi(\boldsymbol{\alpha}^*_{Homo}/\alpha(n_1,n_2)) \\ = &\left(\frac{1}{2}\right)^{2k-1}\left[\beta(n_1+1,n_2)\mathbf{1}\{(n_1+1,n_2)\in S/E\} - \beta(n_1,n_2+1)\mathbf{1}\{(n_1,n_2+1)\in S/E\}\right] \end{aligned}$$

$$+\left(\frac{1}{2}\right)^{2k+1}\left[\beta(n_1-1,n_2)\mathbf{1}\{(n_1-1,n_2)\in S/E\}-\beta(n_1,n_2-1)\mathbf{1}\{(n_1,n_2-1)\in S/E\}\right].$$

Consider $(n_1,n_2)=(0,0)$, $\varphi(\boldsymbol{\alpha}^*_{Homo}/\alpha(0,0))=\left(\frac{1}{2}\right)^{2k-1}[\beta(1,0)-\beta(0,1)]$. As r takes different values over (n_1,n_2), $r(1,0)\neq r(0,1)$ in particular, which implies that $\beta(1,0)\neq\beta(0,1)$ under the biased rule $\boldsymbol{\alpha}^*_{Homo}$ and therefore $\varphi(\boldsymbol{\alpha}^*_{Homo}/\alpha(0,0))\neq 0$. This further implies that $\alpha(0,0)=(v_B/v_A)^{r(0,0)}$ does not maximize $\varphi(\boldsymbol{\alpha}^*_{Homo}/\alpha(0,0))p_A(0,0)+\omega(\boldsymbol{\alpha}^*_{Homo}/\alpha(0,0))\beta(0,0)$. By Proposition 4, the biased rule $\boldsymbol{\alpha}^*_{Homo}$ does not maximize the expected aggregate effort. $\square$

In contrast to Corollary 3, when battles are homogeneous, the biased rule $\boldsymbol{\alpha}^*_{Homo}=\{\alpha(n_1,n_2)|\alpha(n_1,n_2)=\alpha^*_{\text{One-shot}}(r(n_1,n_2))\}$ uniquely maximizes the expected aggregate effort as shown in Theorem 1.

Corollary 4. *Given the biases at states excluding* (n_1,n_2)*, i.e.,* $\boldsymbol{\alpha}/\alpha(n_1,n_2):=\{\alpha(n_1',n_2')$*:* $(n_1',n_2')\in S/(E\cup(n_1,n_2))\}$*,*

(i) if $P((0,0),(n_1,n_2)|\boldsymbol{\alpha}/\alpha(n_1,n_2))>0$ *and* $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))>(<)0$*, the optimal* $\alpha^*(n_1,n_2)>(<)\alpha^*_{One\text{-}shot}(r(n_1,n_2)$*;*

(ii) if $P((0,0),(n_1,n_2)|\boldsymbol{\alpha}/\alpha(n_1,n_2))=0$*, the choice of* $\alpha(n_1,n_2)$ *does not affect the resulting expected aggregate effort.*

Proof. Given $\boldsymbol{\alpha}/\alpha(n_1,n_2)$ such that $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))>0$, to maximize $\mathbf{TE}_{Hete,k}$, the optimal $\alpha^*(n_1,n_2)$ must maximize $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2)$. For $r\in(0,+\infty)$, one can easily verify that $\beta(n_1,n_2)$ increases with $\alpha(n_1,n_2)$ from 0 until $(v_B/v_A)^{r(n_1,n_2)}$. As a result, for any $\alpha(n_1,n_2)\leq(v_B/v_A)^{r(n_1,n_2)}$, an increase in $\alpha(n_1,n_2)$ leads to an increase in $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))p_A(n_1,n_2)+\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))\beta(n_1,n_2)$, since $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))$, $\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2))>0$ and $p_A(n_1,n_2)$ always increases in $\alpha(n_1,n_2)$. Therefore, $\alpha^*(n_1,n_2)\notin[0,(v_B/v_A)^{r(n_1,n_2)}]$, which implies that $\alpha^*(n_1,n_2)>(v_B/v_A)^{r(n_1,n_2)}$. If $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2))<0$, the optimal $\alpha^*(n_1,n_2)<(v_B/v_A)^{r(n_1,n_2)}$, since $\beta(n_1,n_2)$ decreases with $\alpha(n_1,n_2)$ when $\alpha(n_1,n_2)>(v_B/v_A)^{r(n_1,n_2)}$. Analogously, $\beta(n_1,n_2)$ increases with $\alpha(n_1,n_2)$ from 0 until v_B/v_A and decreases with $\alpha(n_1,n_2)$ when $\alpha(n_1,n_2)>v_B/v_A$. The corollary thus follows. $\square$

The condition $P((0,0),(n_1,n_2)|\boldsymbol{\alpha}/\alpha(n_1,n_2))>0$ means that the state (n_1,n_2) is attainable under the biases $\boldsymbol{\alpha}/\alpha(n_1,n_2)$. When biases can further depend on (n_1,n_2), Corollary 4 provides conditions under which the optimal bias $\alpha^*(n_1,n_2)$ favors team A at state (n_1,n_2). The conditions simply say that if the overall effect of the bias on the total effort is positive (resp. negative), the bias should also be set in the favor of team A (resp. team B) at the optimum.

4.2.3 Examples

We provide five examples in this subsection. The first two examples are used to illustrate the idea of decomposition in the last subsection. Examples 3 and 4 show some numerical results for homogeneous and heterogeneous cases, respectively. In Example 5, we compare the optimal biases of 3-battle team contests to those of 3-battle individual contests studied by Barbieri and Serena (2022).

Example 1 (Continue)

Consider a best-of-three team contest with outcome-dependent heterogeneity. There are two methods to compute the coefficient of $p_A(n_1', n_2')$ in $TE_{Hete,k=1}$, i.e., $\varphi(\boldsymbol{\alpha}/\alpha(n_1', n_2'))$. The first way is to apply Lemma 5 and the second one is to decompose $TE_{Hete,k=1}$ in Example 1 directly. The two yield the same result. The details are relegated into subsection 5.12 in the appendix. We present the results as follows.

$$
\begin{aligned}
\varphi(\boldsymbol{\alpha}/\alpha(0,0)) &= p_B(1,1)\beta(1,0) - p_A(1,1)\beta(0,1) + (p_B(1,0) - p_A(0,1))\,\beta(1,1). \\
\varphi(\boldsymbol{\alpha}/\alpha(1,0)) &= p_B(1,1)\beta\,(0,0) - p_A(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(0,1)) &= -p_A(1,1)\beta\,(0,0) + p_B(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(1,1)) &= [-p_A(1,0) + p_B(0,1)]\,\beta\,(0,0) - p_A(0,0)\beta\,(1,0) + p_B(0,0)\beta\,(0,1)\,.
\end{aligned}
$$

In addition, using $\omega(\boldsymbol{\alpha}/\alpha(n_1,n_2)) := \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)$, we have

$$
\begin{aligned}
\omega(\boldsymbol{\alpha}/\alpha(0,0)) &= \sum_{g\in G((0,0),(1,1))} \widehat{P}(g) = p_A(1,0)p_B(1,1) + p_B(0,1)p_A(1,1); \\
\omega(\boldsymbol{\alpha}/\alpha(1,0)) &= p_A(0,0)p_B(1,1); \\
\omega(\boldsymbol{\alpha}/\alpha(0,1)) &= p_B(0,0)p_A(1,1); \\
\omega(\boldsymbol{\alpha}/\alpha(1,1)) &= \sum_{g\in G((0,0),(1,1))} P(g) = p_A(0,0)p_B(1,0) + p_B(0,0)p_A(0,1).
\end{aligned}
$$

From Corollary 2, when $r(n_1, n_2) \leq 1$, the optimal biases can be obtained by solved from the following system:

$$
\begin{aligned}
\left[\begin{array}{c} p_B(1,1)\beta(1,0) - p_A(1,1)\beta(0,1) \\ + \left(p_B(1,0) - p_A(0,1)\right)\beta(1,1) \end{array}\right] \frac{dp_A(0,0)}{d\alpha(0,0)} + \left[\begin{array}{c} p_A(1,0)p_B(1,1) \\ +p_B(0,1)p_A(1,1) \end{array}\right] \frac{d\beta(0,0)}{d\alpha(0,0)} &= 0; \\
\left[p_B(1,1)\beta(0,0) - p_A(0,0)\beta(1,1)\right] \frac{dp_A(1,0)}{d\alpha(1,0)} + p_A(0,0)p_B(1,1)\frac{d\beta(1,0)}{d\alpha(1,0)} &= 0; \\
\left[-p_A(1,1)\beta(0,0) + p_B(0,0)\beta(1,1)\right] \frac{dp_A(0,1)}{d\alpha(0,1)} + p_B(0,0)p_A(1,1)\frac{d\beta(0,1)}{d\alpha(0,1)} &= 0; \\
\left[\begin{array}{c} \left(-p_A(1,0) + p_B(0,1)\right)\beta(0,0) \\ -p_A(0,0)\beta(1,0) + p_B(0,0)\beta(0,1) \end{array}\right] \frac{dp_A(1,1)}{d\alpha(1,1)} + \left[\begin{array}{c} p_A(0,0)p_B(1,0) \\ +p_B(0,0)p_A(0,1) \end{array}\right] \frac{d\beta(1,1)}{d\alpha(1,1)} &= 0.
\end{aligned}
$$

Example 2 (Continue)

In a best-of-five team contest with outcome-dependent heterogeneity, consider state $(1,2)$ for example. We apply the aforementioned two approaches to derive $\varphi(\boldsymbol{\alpha}/\alpha(1,2))$ and $\omega(\boldsymbol{\alpha}/\alpha(1,2))$. By direct comparison, the two approaches generate the same result. We present the results as follows. Details are relegated into subsection 5.13 in the appendix.

$$
\begin{aligned}
\varphi(\boldsymbol{\alpha}/\alpha(1,2)) = \; & \left(p_B(0,1)p_B(0,2) - p_A(1,0)p_B(1,1) - p_B(0,1)p_A(1,1)\right)p_A(2,2)\beta(0,0) \\
& -P(1,0)p_B(1,1)p_A(2,2)\beta(1,0) \\
& +P(0,1)\left(p_B(0,2) - p_A(1,1)\right)p_A(2,2)\beta(0,1) \\
& +P(0,2)p_A(2,2)\beta(0,2) - P(1,1)p_A(2,2)\beta(1,1) \\
& +P(1,2)\beta(2,2),
\end{aligned}
$$

and

$$\omega(\boldsymbol{\alpha}/\alpha(1,2)) := P(1,2)p_A(2,2).$$

Moreover,

$$TE_{Hete,k=2}(\boldsymbol{\alpha}, \mathbf{r}) = \varphi(\boldsymbol{\alpha}/\alpha(1,2))p_A(1,2) + \omega(\boldsymbol{\alpha}/\alpha(1,2))\beta(1,2) + T(\boldsymbol{\alpha}/\alpha(1,2)),$$

where $T(\boldsymbol{\alpha}/\alpha(1,2))$ denotes terms that do not involve $\alpha(1,2)$.

Example 3 (Three-battle contest with homogeneous battles)

Consider a best-of-three team contest with homogeneous battles. For homogeneous battles, we mean that r and α remain the same across the states while two teams' values of the prize (v_A and v_B) can be different. Suppose that team A's value of the prize is v_A and team B's value $v_B = 1$. Let r and $v_A = 0.5$; 1; 1.5; 2; $+\infty$ (all-pay contest), respectively. By Theorem 1, $\alpha^* = (v_B/v_A)^r$ for $r \in (0, +\infty)$ and $\alpha^* = v_B/v_A$ for $r = +\infty$ are the corresponding optimal biased rule. The results are summarized in the following table.

r	$v_A = 0.5$	$v_A = 1$	$v_A = 1.5$	$v_A = 2$
0.5	1.414	1	0.817	0.707
1	2	1	0.667	0.5
1.5	2.828	1	0.544	0.354
2	4	1	0.444	0.25
$+\infty$	2	1	0.667	0.5

Table 1: Optimal Bias for Homogeneous Battles

Example 4 (Three-battle contest with outcome-dependent heterogeneity)

In this example, we will introduce outcome-dependent heterogeneity into a best-of-three contest to see how the optimal biases react. In addition, we will use the example to verify our general property in Corollary 4. To do so, we fix $r(0,0) = 1$, $r(0,1) = 1$, $r(1,1) = 1$, and vary $r(1,0)$ to see how the optimal biased rule $\boldsymbol{\alpha}^* = \{\alpha^*(0,0), \alpha^*(1,0), \alpha^*(0,1), \alpha^*(1,1)\}$ responses to different value of $r(1,0)$. We rely on the general problem (12), or equivalently Proposition 4 to compute the optimal biases numerically. The results are summarized in the following table.

r(1,0)	$\alpha^*(0,0)$	$\alpha^*(1,0)$	$\alpha^*(0,1)$	$\alpha^*(1,1)$	TE
0.5	0.667	1	1	1.5	0.7
1	1	1	1	1	0.75
1.5	2	1	1	0.5	0.833
any value$\in [2, +\infty]$	$+\infty$	1	any value	0	1

Table 2: Optimal Biases For Heterogeneous Case

For $r(1,0) \in [2, +\infty]$, $\alpha^*(0,0) = +\infty$ means that team A wins battle 1 automatically. As a consequence, in battle 2, the state must be $(1,0)$, at which players compete for a prize of value 1 in a Tullock contest with $r(1,0) \in [2, +\infty]$. This is because at $(1,0)$, if player $A(2)$

wins battle 2, team A wins the contest; otherwise, team B wins the contest, as $\alpha^*(1,1) = 0$. The reason why $\alpha^*(0,1)$ can be any value is that state $(0,1)$ is never attainable.

To verify the property in Corollary 4, we compute the winning chances under the optimal biased rule as below.

$r(1,0)$	$p_A(0,0)$	$p_A(1,0)$	$p_A(0,1)$	$p_A(1,1)$
0.5	0.4	0.5	0.5	0.6
1	0.5	0.5	0.5	0.5
1.5	0.6667	0.5	0.5	0.3333
any value$\in [2, +\infty]$	1	0.5	[0,1]	0

and

$r(1,0)$	$\beta(0,0)$	$\beta(1,0)$	$\beta(0,1)$	$\beta(1,1)$
0.5	0.48	0.25	0.5	0.48
1	0.5	0.5	0.5	0.5
1.5	0.4444	0.6667	0.5	0.4444
$[2, +\infty]$	0	1	$\beta(1, 1, \alpha^*(0,1), r(1,0))$	0

Recall that in Example 1 (Continue), we have

$$\begin{aligned}
\varphi(\boldsymbol{\alpha}/\alpha(0,0)) &= p_B(1,1)\beta(1,0) - p_A(1,1)\beta(0,1) + (p_B(1,0) - p_A(0,1))\,\beta(1,1); \\
\varphi(\boldsymbol{\alpha}/\alpha(1,0)) &= p_B(1,1)\beta\,(0,0) - p_A(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(0,1)) &= -p_A(1,1)\beta\,(0,0) + p_B(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(1,1)) &= [-p_A(1,0) + p_B(0,1)]\,\beta\,(0,0) - p_A(0,0)\beta\,(1,0) + p_B(0,0)\beta\,(0,1)\,.
\end{aligned}$$

By direct substitution, we could compute $\varphi(\boldsymbol{\alpha}^*/\alpha(n_1,n_2))$ for all attainable states, i.e., $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ in this example. The results are summarized in the following table.

Corollary 4(i) says that for the attainable states, if $\varphi(\boldsymbol{\alpha}^*/\alpha(n_1,n_2)) > (<)0$, the optimal $\alpha^*(n_1,n_2) > (<)\,(v_B/v_A)^{r(n_1,n_2)}$ and if $\varphi(\boldsymbol{\alpha}/\alpha(n_1,n_2)) = 0$, $\alpha^*(n_1,n_2) = (v_B/v_A)^{r(n_1,n_2)}$ for $r(n_1,n_2) \in (0,+\infty)$. In this example, $v_B/v_A = 1$ and one could easily verify the property for $r(1,0) = 0.5$; 1; 1.5. When $r(1,0) \in [2,+\infty)$, the property remains valid for state $(0,0)$, $(1,0)$, $(1,1)$. For $(0,1)$, the condition of Corollary 4(i) is violated, since $p_A(0,0) =$

$r(1,0)$	$\varphi(\boldsymbol{\alpha}^*/\alpha(0,0))$	$\varphi(\boldsymbol{\alpha}^*/\alpha(1,0))$	$\varphi(\boldsymbol{\alpha}^*/\alpha(0,1))$	$\varphi(\boldsymbol{\alpha}^*/\alpha(1,1))$
0.5	-0.2	0	0	0.2
1	0	0	0	0
1.5	0.2778	0	0	-0.2778
$[2,+\infty]$	1	0	0	-1

Table 3: $\varphi(\boldsymbol{\alpha}^*/\alpha(n_1,n_2))$ in Example 4

1 under the optimal biased rule and the contest would never reach the state $(0,1)$, i.e., $P((0,0),(0,1)|\boldsymbol{\alpha}^*)=0$. In this case, it follows from Corollary 4(ii) that $\alpha^*(0,1)\in[0,+\infty)$.[29]

In the following example, we compare our results to Barbieri and Serena (2022) who study multi-battle individual contests.

Examples 5 (Team contests v.s. individual contests)

Barbieri and Serena (2022) consider a best-of-three Tullock contest between two ex-ante symmetric players. In particular, they study both victory-dependent biases and victory-independent biases that maximize the expected total effort. To compare, we let $v_A=v_B$ and r remain the same across the states.

(i) Consider biases are victory-dependent. In a three-battle contest between two teams, our Example 4 shows that $\{\alpha(0,0),\ \alpha(1,0),\ \alpha(0,1),\ \alpha(1,1)\}=\{1,\ 1,\ 1,\ 1\}$ maximizes the expected total effort when $r=1$ across battles and two matched players have equal chance to win a battle at each node.

In a three-battle contest with two individuals, Proposition 1 of Barbieri and Serena (2022) shows that $\{\alpha(0,0),\ \alpha(1,0),\ \alpha(0,1),\ \alpha(1,1)\}=\{1,\ 1/3,\ 3,\ 1\}$ is the unique global maximum for the expected total effort when $r=1$ across battles. In particular, players equally likely to win a battle at each node.

(ii) Consider victory-independent biases. In this case, battles are homogeneous in our setting. By Theorem 1, the optimal bias is $\alpha^*=1$, as a result, two matched players are equally likely to win each battle in our team contest setting.

In contrast, Proposition 2 of Barbieri and Serena (2022) shows that the fully unbiased contest is not an optimal victory-independent contest, i.e., two players do not have equal chance to win each battle.

[29] If one applies Corollary 4(i) to conclude that $\varphi(\boldsymbol{\alpha}/\alpha(0,1))=0$ leads to $\alpha^*(0,1)=1$. Note that $\alpha^*(0,1)=1$ with $\alpha^*(0,0)=+\infty$, $\alpha^*(1,0)=1$, and $\alpha^*(1,1)=0$ still constitutes a specifical optimal biased rule.

5 Concluding Remarks

We study the effort-maximizing biased design of a dynamic team contest with pairwise battles in which players from two rival teams form pairwise matches and compete head-to-head in their own battles. Our model is inherited from Fu, Lu, and Pan (2015) but differs from their model in two aspects: Two teams can be asymmetric and outcome-dependent heterogeneity is introduced. With two asymmetric teams, we show that two paired players always have a common prize spread, regardless of the previous battle outcomes; while, the result of history independent in Fu, Lu, and Pan (2015) is no longer valid for a biased team contest with outcome-dependent heterogeneity.

The best-of-N contest studied here is of arbitrary odd length and battles are played sequentially. The designer can impose a treatment to bias each battle contest. The effort-maximizing biases are as follows. When battles are homogeneous, the full-balance rule is effort-maximizing, which echoes the previous finding in static contests: Levelling the playing field maximizes the competition. However, this full-balance rule is in general suboptimal in eliciting efforts in the presence of outcome-dependent heterogeneity. Our result domonstrates that outcome-dependent heterogeneity is crucial in designing the optimal biases, which offers a new insight for dynamic contest design problems.

In this paper, we attempt to shed light on the optimal biased design of dynamic team contests. For that, we adopt this multi-battle team contest setting as the first step. A key feature of this setting is that momentum effect is absent, which allows us to consider contests of arbitrary odd length, generalized contest technology, and outcome-dependent heterogeneity. Our design aims to maximize the performance of the competing agents, specifically the expected total effort. Many works, such as Barbieri and Serena (2019) and Barbieri and Serena (2022), also consider the goal of maximizing the winner's effort. However, aggregating the winner's effort and identifying the general properties would be complicated by allowing arbitrary odd battles and outcome-dependent heterogeneity. We leave these to future work.

Appendix

This appendix covers the proofs omitted in the main text.

5.1 Proof of Property 1

We consider two cases: (1) $r \in (0, +\infty)$ and (2) $r = +\infty$, i.e., all-pay auction.

Case 1: For any $r \in (0, +\infty)$, a biased Tullock contest $(\Delta u_{A(t)}, \Delta u_{B(t)}, c_{A(t)} = 1, c_{B(t)} = 1, \alpha)$ is equivalent to a non-biased one $(\alpha^{1/r}\Delta u_{A(t)}, \Delta u_{B(t)}, c_{A(t)} = 1, c_{B(t)} = 1, \alpha = 1)$. Denote the resulting equilibrium effort $(x_{A(t)}, x_{B(t)})$ and $(\widetilde{x}_{A(t)}, x_{B(t)})$, respectively.

To see the equivalency, player $A(t)$ chooses effort $x_{A(t)}$ to maximize

$$\frac{\alpha x_{A(t)}^r}{\alpha x_{A(t)}^r + x_{B(t)}^r}\Delta u_{A(t)} - x_{A(t)}$$

$$= \frac{\widetilde{x}_{A(t)}^r}{\widetilde{x}_{A(t)}^r + x_{B(t)}^r}\Delta u_{A(t)} - \frac{\widetilde{x}_{A(t)}}{\alpha^{1/r}},$$

is equivalent to

$$\frac{\widetilde{x}_{A(t)}^r}{\widetilde{x}_{A(t)}^r + x_{B(t)}^r}\alpha^{1/r}\Delta u_{A(t)} - \widetilde{x}_{A(t)}.$$

We therefore conclude that $\widetilde{x}_{A(t)} = \alpha^{1/r}x_{A(t)}$. As a result, the total effort in battle t equals $E\left[x_{A(t)} + x_{B(t)}\right] = E\left[\widetilde{x}_{A(t)}/\alpha^{1/r} + x_{B(t)}\right]$. To derive the equilibrium effort, consider a two-player one-shot Tullock contest with an arbitrary $r(t)$, the equilibrium strategies are summarized in Lemma 1 in Feng and Lu (2018), by which we solve for the equilibrium effort $(x_{A(t)}, x_{B(t)})$ for a non-biased Tullock contest, $(\alpha^{1/r}\Delta u_{A(t)}, \Delta u_{B(t)}, c_{A(t)} = 1, c_{B(t)} = 1, \alpha = 1)$. By applying the equivalency result, we characterize the equilibrium effort $(\widetilde{x}_{A(t)}, x_{B(t)})$ and compute the resulting total effort. Let $\widehat{r}(z) \in (1, 2)$ represent the unique solution to $r = 1 + z^r$ with $z \in (0, 1]$. The results are summarized as follows.

Case 1.1: If $\alpha^{1/r}\Delta u_{A(t)} \geq \Delta u_{B(t)}$,

(i) If $r \leq \hat{r}(\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}})$,

$$\widetilde{x}_{A(t)} = \frac{r\left(\alpha^{1/r}\Delta u_{A(t)}\right)^{r+1}(\Delta u_{B(t)})^r}{[\left(\alpha^{1/r}\Delta u_{A(t)}\right)^r + (\Delta u_{B(t)})^r]^2}, x_{B(t)} = \frac{r\left(\alpha^{1/r}\Delta u_{A(t)}\right)^r(\Delta u_{B(t)})^{r+1}}{[\left(\alpha^{1/r}\Delta u_{A(t)}\right)^r + (\Delta u_{B(t)})^r]^2}.$$

Therefore,

$$\begin{aligned}
& E\left[x_{A(t)}+x_{B(t)}\right] \\
= & \frac{r\alpha\left(\Delta u_{A(t)}\right)^{r}(\Delta u_{B(t)})^{r}\left[\Delta u_{A(t)}+\Delta u_{B(t)}\right]}{[\alpha\left(\Delta u_{A(t)}\right)^{r}+(\Delta u_{B(t)})^{r}]^{2}} \\
= & \frac{r\alpha v_{A}^{r}v_{B}^{r}}{\left(\alpha v_{A}^{r}+v_{B}^{r}\right)^{2}}\left(v_{A}+v_{B}\right)\Delta u(t).
\end{aligned}$$

using $\Delta u_{A(t)}=v_{A}\Delta u(t)$ and $\Delta u_{B(t)}=v_{B}\Delta u(t)$.
(ii) If $r\in(\hat{r}(\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}}),2]$,

$$\begin{aligned}
\widetilde{x}_{A(t)} &= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\Delta u_{B(t)}; \\
x_{B(t)} &= \begin{cases} (1-\frac{1}{r})\Delta u_{B(t)}, & \text{with probability } q=\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \text{with probability } 1-q. \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
E\left[x_{A(t)}+x_{B(t)}\right] &= E\left[\widetilde{x}_{A(t)}/\alpha^{1/r}+x_{B(t)}\right] \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\frac{\Delta u_{B(t)}}{\alpha^{1/r}}+(\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\frac{\Delta u_{B(t)}}{\alpha^{1/r}}\cdot\frac{\Delta u_{B(t)}}{\Delta u_{A(t)}} \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\frac{\Delta u_{B(t)}}{\alpha^{1/r}}\left(1+\frac{\Delta u_{B(t)}}{\Delta u_{A(t)}}\right) \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\frac{v_{B}}{\alpha^{1/r}}\left(1+\frac{v_{B}}{v_{A}}\right)\Delta u(t).
\end{aligned}$$

(iii) If $r>2$,

$$\widetilde{x}_{A(t)}=\mu^{*},\ x_{B(t)}=\begin{cases} \mu^{*}, & \text{with probability } q=\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}}, \\ 0, & \text{with probability } 1-q. \end{cases}$$

Therefore,

$$
\begin{aligned}
E\left[x_{A(t)}+x_{B(t)}\right] &= E\left[\widetilde{x}_{A(t)}/\alpha^{1/r}+x_{B(t)}\right] \\
&= \frac{\Delta u_{B(t)}}{2\alpha^{1/r}}+\frac{\Delta u_{B(t)}}{2}\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}} \\
&= \frac{\Delta u_{B(t)}}{2\alpha^{1/r}}\left(1+\frac{\Delta u_{B(t)}}{\Delta u_{A(t)}}\right) \\
&= \frac{1}{2\alpha^{1/r}}\left(1+\frac{v_B}{v_A}\right)v_B\Delta u(t).
\end{aligned}
$$

Case 1.2: If $\Delta u_{B(t)} \geq \alpha^{1/r}\Delta u_{A(t)}$,
(i) If $r \leq \hat{r}(\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}})$,

$$
\widetilde{x}_{A(t)}=\frac{r\left(\alpha^{1/r}\Delta u_{A(t)}\right)^{r+1}(\Delta u_{B(t)})^r}{[\left(\alpha^{1/r}\Delta u_{A(t)}\right)^r+(\Delta u_{B(t)})^r]^2}, x_{B(t)}=\frac{r\left(\alpha^{1/r}\Delta u_{A(t)}\right)^{r}(\Delta u_{B(t)})^{r+1}}{[\left(\alpha^{1/r}\Delta u_{A(t)}\right)^r+(\Delta u_{B(t)})^r]^2}.
$$

Therefore,

$$
\begin{aligned}
& E\left[x_{A(t)}+x_{B(t)}\right] \\
=& \frac{r\alpha\left(\Delta u_{A(t)}\right)^r(\Delta u_{B(t)})^r\left[\Delta u_{A(t)}+\Delta u_{B(t)}\right]}{[\alpha\left(\Delta u_{A(t)}\right)^r+(\Delta u_{B(t)})^r]^2} \\
=& \frac{r\alpha v_A^r v_B^r}{\left(\alpha v_A^r+v_B^r\right)^2}\left(v_A+v_B\right)\Delta u(t).
\end{aligned}
$$

(ii) If $r \in (\hat{r}(\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}), 2]$,

$$
\begin{aligned}
x_{B(t)} &= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\alpha^{1/r}\Delta u_{A(t)}; \\
\widetilde{x}_{A(t)} &= \begin{cases} (1-\frac{1}{r})\alpha^{1/r}\Delta u_{A(t)}, & \text{with probability } q=\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \text{with probability } 1-q. \end{cases}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
E\left[x_{A(t)}+x_{B(t)}\right] &= E\left[\widetilde{x}_{A(t)}/\alpha^{1/r}+x_{B(t)}\right] \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\Delta u_{A(t)}\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}+(\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\alpha^{1/r}\Delta u_{A(t)} \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\alpha^{1/r}\Delta u_{A(t)}\left(1+\frac{\Delta u_{A(t)}}{\Delta u_{B(t)}}\right) \\
&= (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\alpha^{1/r}\left(1+\frac{v_A}{v_B}\right)v_A v.
\end{aligned}$$

(iii) If $r>2$,

$$x_{B(t)}=\mu^*,\ \widetilde{x}_{A(t)}=\begin{cases}\mu^*, & \text{with probability } q=\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}},\\ 0, & \text{with probability } 1-q.\end{cases}$$

Therefore,

$$\begin{aligned}
E\left[x_{A(t)}+x_{B(t)}\right] &= E\left[\widetilde{x}_{A(t)}/\alpha^{1/r}+x_{B(t)}\right] \\
&= \frac{\Delta u_{A(t)}}{2}\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}+\frac{\alpha^{1/r}\Delta u_{A(t)}}{2} \\
&= \frac{\alpha^{1/r}\Delta u_{A(t)}}{2}\left[\frac{\Delta u_{A(t)}}{\Delta u_{B(t)}}+1\right] \\
&= \frac{\alpha^{1/r}}{2}\left(1+\frac{v_A}{v_B}\right)v_A\Delta u(t).
\end{aligned}$$

Case 2: For $r=+\infty$, a biased Tullock contest $(\Delta u_{A(t)},\Delta u_{B(t)},c_{A(t)}=1,c_{B(t)}=1,\alpha)$ is equivalent to a non-biased one $(\alpha\Delta u_{A(t)},\Delta u_{B(t)},c_{A(t)}=1,c_{B(t)}=1,\alpha=1)$. Denote the resulting equilibrium effort $(x_{A(t)},x_{B(t)})$ and $(\widetilde{x}_{A(t)},x_{B(t)})$, respectively. To see the equivalency, player $A(t)$ chooses effort $x_{A(t)}$ to maximize

$$\Delta u_{A(t)}\Pr(\alpha x_{A(t)}>x_{B(t)})-x_{A(t)},$$

i.e.,

$$\Delta u_{A(t)} \Pr(\alpha x_{A(t)} > x_{B(t)}) - \frac{1}{\alpha}\alpha x_{A(t)},$$

which is equivalent to maximizing

$$\alpha \Delta u_{A(t)} \Pr(\alpha x_{A(t)} > x_{B(t)}) - \alpha x_{A(t)},$$

or

$$\alpha \Delta u_{A(t)} \Pr(\widetilde{x}_{A(t)} > x_{B(t)}) - \widetilde{x}_{A(t)}.$$

We therefore conclude that $\widetilde{x}_{A(t)} = \alpha x_{A(t)}$. As a result, the total effort in battle t equals $E\left[x_{A(t)} + x_{B(t)}\right] = E\left[\widetilde{x}_{A(t)}/\alpha + x_{B(t)}\right]$.

Case 2.1: If $\alpha \Delta u_{A(t)} \geq \Delta u_{B(t)}$,

$$\widetilde{x}_{A(t)} = \mu^*,\ x_{B(t)} = \begin{cases} \mu^*, & \text{with probability } q = \frac{\Delta u_{B(t)}}{\alpha \Delta u_{A(t)}}, \\ 0, & \text{with probability } 1-q. \end{cases}$$

Therefore,

$$\begin{aligned} E\left[x_{A(t)} + x_{B(t)}\right] &= E\left[\widetilde{x}_{A(t)}/\alpha + x_{B(t)}\right] \\ &= \frac{\Delta u_{B(t)}}{2\alpha} + \frac{\Delta u_{B(t)}}{2} \frac{\Delta u_{B(t)}}{\alpha \Delta u_{A(t)}} \\ &= \frac{\Delta u_{B(t)}}{2\alpha}\left(1 + \frac{\Delta u_{B(t)}}{\Delta u_{A(t)}}\right) \\ &= \frac{1}{2\alpha}\left(1 + \frac{v_B}{v_A}\right) v_B \Delta u(t). \end{aligned}$$

Case 2.2: If $\Delta u_{B(t)} \geq \alpha \Delta u_{A(t)}$,

$$x_{B(t)} = \mu^*,\ \widetilde{x}_{A(t)} = \begin{cases} \mu^*, & \text{with probability } q = \frac{\alpha \Delta u_{A(t)}}{\Delta u_{B(t)}}, \\ 0, & \text{with probability } 1-q. \end{cases}$$

Therefore,

$$E\left[x_{A(t)} + x_{B(t)}\right] = E\left[\widetilde{x}_{A(t)}/\alpha + x_{B(t)}\right]$$

$$
\begin{aligned}
&= \frac{\Delta u_{A(t)}}{2}\frac{\alpha\Delta u_{A(t)}}{\Delta u_{B(t)}} + \frac{\alpha\Delta u_{A(t)}}{2} \\
&= \frac{\alpha\Delta u_{A(t)}}{2}\left[\frac{\Delta u_{A(t)}}{\Delta u_{B(t)}} + 1\right] \\
&= \frac{\alpha}{2}\left(1 + \frac{v_A}{v_B}\right) v_A \Delta u(t).
\end{aligned}
$$

In sum, for $r(t) \in (0, +\infty)$, when $\alpha^{1/r}\Delta u_{A(t)} \geq \Delta u_{B(t)}$, the analytical formulas of β_t is

$$
\beta_t = \begin{cases} \frac{r\alpha v_A^r v_B^r}{\left(\alpha v_A^r + v_B^r\right)^2}\left(v_A + v_B\right) & \text{if} \quad r(t) \leq \widehat{r}(\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}}), \\ (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\frac{v_B}{\alpha^{1/r}}\left(1+\frac{v_B}{v_A}\right) & \text{if} \quad r(t) \in (\widehat{r}(\frac{\Delta u_{B(t)}}{\alpha^{1/r}\Delta u_{A(t)}}), 2], \\ \frac{1}{2\alpha^{1/r}}\left(1+\frac{v_B}{v_A}\right) v_B & \text{if} \quad r(t) \in (2, +\infty), \end{cases}
$$

when $\Delta u_{B(t)} \geq \alpha^{1/r}\Delta u_{A(t)}$, the analytical formulas of β_t is

$$
\beta_t = \begin{cases} \frac{r\alpha v_A^r v_B^r}{\left(\alpha v_A^r + v_B^r\right)^2}\left(v_A + v_B\right) & \text{if} \quad r(t) \leq \widehat{r}(\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}), \\ (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})\alpha^{1/r}\left(1+\frac{v_A}{v_B}\right) v_A & \text{if} \quad r(t) \in (\widehat{r}(\frac{\alpha^{1/r}\Delta u_{A(t)}}{\Delta u_{B(t)}}), 2], \\ \frac{\alpha^{1/r}}{2}\left(1+\frac{v_A}{v_B}\right) v_A & \text{if} \quad r(t) \in (2, +\infty), \end{cases}
$$

For $r(t) = +\infty$, when $\alpha\Delta u_{A(t)} \geq \Delta u_{B(t)}$, the analytical formula of $\beta_t = \frac{1}{2\alpha}\left(1 + \frac{v_B}{v_A}\right) v_B$; when $\Delta u_{B(t)} \geq \alpha\Delta u_{A(t)}$, the analytical formula of $\beta_t = \frac{\alpha}{2}\left(1 + \frac{v_A}{v_B}\right) v_A$.

5.2 Example 1: 3-battle contest with homogeneous battles

Recall that $V_A(n_1, n_2)$ (resp. $V_B(n_1, n_2)$) denotes the continuation value of team A (resp. team B) at state (n_1, n_2). At state (n_1, n_2), player $A(t)$'s effective prize spreads of winning battle t is $\Delta u_A(n_1, n_2) = V_A(n_1 + 1, n_2) - V_A(n_1, n_2 + 1)$ and player $B(t)$'s effective prize spreads of winning battle t is $\Delta u_B(n_1, n_2) = V_B(n_1, n_2 + 1) - V_B(n_1 + 1, n_2)$.

To solve the game backwards, we compute players' effective prize spreads at state $(1, 1)$, $(1, 0)$, $(0, 1)$, $(0, 0)$.

In battle 3, none of players has incentive to make positive effort at state $(2, 0)$ or $(0, 2)$, since a team wins if the team wins two battles out of three. It only remains to consider the state $(1,1)$, at which players' effective prize spreads of winning the third battle are

$$\Delta u_A(1,1) = v_A;\ \Delta u_B(1,1) = v_B.$$

Players' winning probabilities for this battle are

$$p_A(1,1) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r};\ p_B(1,1) = \frac{v_B^r}{\alpha v_A^r + v_B^r}.$$

The resulting effort is $E(1,1) = \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2}$.

In battle 2, at state $(1,0)$, players' effective prize spreads of winning battle 2 are

$$\begin{aligned}
\Delta u_A(1,0) &= V_A(2,0) - V_A(1,1) = V_A(2,0) - p_A(1,1)V_A(2,0) = p_B(1,1)V_A(2,0) = \frac{v_B^r}{\alpha v_A^r + v_B^r} v_A; \\
\Delta u_B(1,0) &= V_B(1,1) = p_B(1,1)V_B(1,2) = \frac{v_B^r}{\alpha v_A^r + v_B^r} v_B.
\end{aligned}$$

Players' winning probabilities for this battle are

$$p_A(1,0) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r};\ p_B(1,0) = \frac{v_B^r}{\alpha v_A^r + v_B^r}.$$

The resulting effort equals

$$E(1,0) = p_B(1,1) \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} = \frac{\alpha r v_A^r v_B^{2r} (v_A + v_B)}{(\alpha v_A^r + v_B^r)^3}.$$

Likewise, at $(0,1)$,

$$\begin{aligned}
\Delta u_A(0,1) &= V_A(1,1) = p_A(1,1)V_A(2,1) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r} v_A; \\
\Delta u_B(0,1) &= V_B(0,2) - V_B(1,1) = V_B(0,2) - p_B(1,1)V_B(1,2) = p_A(1,1)v_B = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r} v_B.
\end{aligned}$$

Players' winning probabilities for this battle are

$$p_A(0,1) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r};\ p_B(0,1) = \frac{v_B^r}{\alpha v_A^r + v_B^r}.$$

The resulting effort equals

$$E(0,1) = p_A(1,1)\frac{\alpha r v_A^r v_B^r\,(v_A+v_B)}{(\alpha v_A^r + v_B^r)^2} = \frac{\alpha^2 r v_A^{2r} v_B^r\,(v_A+v_B)}{(\alpha v_A^r + v_B^r)^3}.$$

In battle 1, we denote $p_A := \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r}$. At $(0,0)$, player $A(1)$'s effective prize spreads of winning battle 1 is

$$\begin{aligned}
& \Delta u_A(0,0) \\
= \;& V_A(1,0) - V_A(0,1) \\
= \;& p_A(1,0)V_A(2,0) + p_B(1,0)V_A(1,1) - [p_A(0,1)V_A(1,1) + p_B(0,1)V_A(0,2)] \\
= \;& p_A v_A + p_B p_A v_A - [p_A p_A v_A] \\
= \;& p_A v_A\,[1 + p_B - p_A] \\
= \;& 2 p_A p_B v_A.
\end{aligned}$$

Analogously, player $B(1)$'s effective prize spreads of winning battle 1 is

$$\Delta u_B(0,0) = 2 p_A p_B v_B.$$

Players' winning probabilities for this battle are

$$p_A(0,0) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r};\; p_B(0,0) = \frac{v_B^r}{\alpha v_A^r + v_B^r}.$$

The resulting effort equals

$$E(0,0) = 2p_A p_B \frac{\alpha r v_A^r v_B^r\,(v_A+v_B)}{(\alpha v_A^r + v_B^r)^2} = 2\frac{\alpha^2 r v_A^{2r} v_B^{2r}\,(v_A+v_B)}{(\alpha v_A^r + v_B^r)^4}.$$

Therefore, the expected total effort

$$TE = E(0,0) + p_A(0,0)E(1,0) + p_B(0,0)E(0,1)$$

$$
\begin{aligned}
&+[p_A(0,1)p_B(0,1)+p_A(1,0)p_B(1,0)]E(1,1)\\
=\;& 2p_Ap_B\frac{\alpha r v_A^r v_B^r\left(v_A+v_B\right)}{(\alpha v_A^r+v_B^r)^2}+p_Ap_B\frac{\alpha r v_A^r v_B^r\left(v_A+v_B\right)}{(\alpha v_A^r+v_B^r)^2}+p_Bp_A\frac{\alpha r v_A^r v_B^r\left(v_A+v_B\right)}{(\alpha v_A^r+v_B^r)^2}\\
&+2p_Ap_B\frac{\alpha r v_A^r v_B^r\left(v_A+v_B\right)}{(\alpha v_A^r+v_B^r)^2}\\
=\;& 6p_Ap_B\frac{\alpha r v_A^r v_B^r\left(v_A+v_B\right)}{(\alpha v_A^r+v_B^r)^2},
\end{aligned}
$$

where $p_A=\frac{\alpha v_A^r}{\alpha v_A^r+v_B^r}$ and $p_B=\frac{v_B^r}{\alpha v_A^r+v_B^r}$.

We provide an alternative way to solve the 3-battle game. In the proof of Property 1, we show that the contest $(\Delta u_A(n_1,n_2),\Delta u_B(n_1,n_2),c_{A(t)}=1,c_{B(t)}=1,\alpha)$ is equivalent to $(\Delta u(n_1,n_2),\Delta u(n_1,n_2),c_{A(t)}=1/\left(\alpha^{1/r(t)}v_A\right),c_{B(t)}=1/v_B,\alpha=1)$, where $\Delta u(n_1,n_2):=\Delta u_{A(t)}(n_1,n_2)/v_A=\Delta u_{B(t)}(n_1,n_2)/v_B$. Hence, when the state is (n_1,n_2), the effort in battle t equals

$$
\mathbb{E}\left(x_{A(t)}+x_{B(t)}\right)(n_1,n_2)=\frac{\alpha r v_A^r v_B^r}{\left(\alpha v_A^r+v_B^r\right)^2}\left(v_A+v_B\right)\Delta u(n_1,n_2).
$$

Next, we apply the above formula to derive the effort in each battle as follows.

In battle 3, at state $(1,1)$, since $\Delta u_A(1,1)=v_A$ and $\Delta u_B(1,1)=v_B$, $\frac{\Delta u_A(n_1,n_2)}{v_A}=1$ and $E(1,1)=\frac{\alpha r v_A^r v_B^r(v_A+v_B)}{(\alpha v_A^r+v_B^r)^2}$.

In battle 2, at state $(1,0)$, since $\Delta u_A(1,0)=p_A(1,0)\Delta u_A(2,0)+p_B(1,0)\Delta u_A(1,1)=p_Bv_A$, we have $\frac{\Delta u_A(n_1,n_2)}{v_A}=p_B$ and $E(1,0)=\frac{\alpha r v_A^r v_B^r(v_A+v_B)}{(\alpha v_A^r+v_B^r)^2}p_B$.

At state $(0,1)$, since $\Delta u_A(0,1)=p_A(0,1)\Delta u_A(1,1)+p_B(0,1)\Delta u_A(0,2)=p_Av_A$, $\frac{\Delta u_A(n_1,n_2)}{v_A}=p_A$ and $E(0,1)=\frac{\alpha r v_A^r v_B^r(v_A+v_B)}{(\alpha v_A^r+v_B^r)^2}p_A$.

In battle 1, at $(0,0)$, since $\Delta u_A(0,0)=p_A(0,0)\Delta u_A(1,0)+p_B(0,0)\Delta u_A(0,1)=p_Ap_Bv_A+p_Bp_Av_A=2p_Ap_Bv_A$, $\frac{\Delta u_A(n_1,n_2)}{v_A}=2p_Ap_B$ and $E(0,0)=\frac{\alpha r v_A^r v_B^r(v_A+v_B)}{(\alpha v_A^r+v_B^r)^2}2p_Ap_B$.

The expected aggregate effort thus equals

$$
\begin{aligned}
TE\;=\;& E(0,0)\\
&+p_A(0,0)E(1,0)+p_B(0,0)E(0,1)\\
&+[p_A(0,1)p_B(0,1)+p_A(1,0)p_B(1,0)]E(1,1)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} 2 p_A p_B \\
&\quad + \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} p_A p_B + \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} p_A p_B \\
&\quad + \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} 2 p_A p_B \\
&= \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} 6 p_A p_B.
\end{aligned}
$$

5.3 Example 1: 3-battle contest with outcome-dependent heterogeneity

We solve the game backwards. By Property 1, at state (n_1, n_2), the effort in current battle equals $E(n_1, n_2) = \beta(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2)) \Delta u(n_1, n_2)$. We apply the formula to compute the effort in each battle at each possible state as follows.

In battle 3, at state $(1, 1)$, since

$$\Delta u_A(1,1) = v_A; \ \Delta u_B(1,1) = v_B,$$

we have $\Delta u(1,1) = 1$. Hence,

$$E(1,1) = \beta(1,1) \Delta u(1,1) = \beta(1,1),$$

where $\beta(1,1) = \beta(v_A, v_B, \alpha(1,1), r(1,1))$.

In addition, one could verify that $p_A(1,1) = \frac{\alpha(1,1) v_A^{r(1,1)}}{\alpha(1,1) v_A^{r(1,1)} + v_B^{r(1,1)}}$ and $p_B(1,1) = \frac{v_B^{r(1,1)}}{\alpha(1,1) v_A^{r(1,1)} + v_B^{r(1,1)}}$.

In battle 2, at state $(1, 0)$, since

$$
\begin{aligned}
\Delta u_A(1,0) &= p_A(2,0) \Delta u_A(2,0) + p_B(1,1) \Delta u_A(1,1) = p_B(1,1) v_A; \\
\Delta u_B(1,0) &= p_A(2,0) \Delta u_B(2,0) + p_B(1,1) \Delta u_B(1,1) = p_B(1,1) v_B,
\end{aligned}
$$

where $p_B(1,1) = \frac{v_B^{r(1,1)}}{\alpha(1,1) v_A^{r(1,1)} + v_B^{r(1,1)}}$. Hence,

$$E(1,0) = \beta(1,0) \Delta u(1,0) = \beta(1,0) p_B(1,1),$$

where $\beta(1,0) = \beta(v_A, v_B, \alpha(1,0), r(1,0))$ and $\Delta u(1,0) = p_B(1,1) = \frac{v_B^{r(1,1)}}{\alpha(1,1)v_A^{r(1,1)} + v_B^{r(1,1)}}$.

At state $(0,1)$, we have

$$\begin{aligned}\Delta u_A(0,1) &= p_A(1,1)\Delta u_A(1,1) + p_B(0,2)\Delta u_A(0,2) = p_A(1,1)\Delta u_A(1,1) = p_A(1,1)v_A; \\ \Delta u_B(0,1) &= p_A(1,1)\Delta u_B(1,1) + p_B(0,2)\Delta u_B(0,2) = p_A(1,1)v_B;\end{aligned}$$

where $p_A(1,1) = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r}$. Hence,

$$E(0,1) = \beta(0,1)\,\Delta u(0,1) = \beta(0,1)\,p_A(1,1),$$

where $\beta(0,1)$ and $\Delta u(0,1) = p_A(1,1)$.

In battle 1, the state is $(0,0)$, we have

$$\begin{aligned}\Delta u_A(0,0) &= p_A(1,0)\Delta u_A(1,0) + p_B(0,1)\Delta u_A(0,1) \\ &= p_A(1,0)p_B(1,1)v_A + p_B(0,1)p_A(1,1)v_A,\end{aligned}$$

which gives

$$\Delta u(0,0) = \Delta u_A(0,0)/v_A = p_A(1,0)p_B(1,1) + p_B(0,1)p_A(1,1).$$

Hence, the total effort in battle 1 equals

$$E(0,0) = \beta(0,0)\,\Delta u(0,0).$$

Therefore, the expected aggregate effort equals

$$\begin{aligned}TE &= E(0,0) + p_A(0,0)E(1,0) + p_B(0,0)E(0,1) \\ &\quad + [p_A(0,0)p_B(1,0) + p_B(0,0)p_A(0,1)]E(1,1) \\ &= [p_A(1,0)p_B(1,1) + p_B(0,1)p_A(1,1)]\,\beta(0,0) \\ &\quad + p_A(0,0)p_B(1,1)\beta(1,0) + p_B(0,0)p_A(1,1)\beta(0,1) \\ &\quad + [p_A(0,0)p_B(1,0) + p_B(0,0)p_A(0,1)]\beta(1,1).\end{aligned}$$

5.4 Example 2: 5-battle contest with homogeneous battles

Consider a 5-battle contest between two teams with pairwise battles. The winner is the team with marjority wins. By Property 1, at state (n_1, n_2), the effort in current battle equals

$$\mathbb{E}\left(x_{A(t)} + x_{B(t)}\right)(n_1, n_2) = \frac{\alpha r v_A^r v_B^r}{\left(\alpha v_A^r + v_B^r\right)^2}\left(v_A + v_B\right)\frac{\Delta u_A(n_1, n_2)}{v_A}.$$

We apply the above formula to solve the contest game backwards.

In battle 5, at $(2,2)$, we have $\Delta u_A(2,2) = v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = 1$. By applying the formula in Property 1, $E(1,1) = \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

In battle 4, at $(2,1)$, we have $\Delta u_A(2,1) = p_A(2,1)\Delta u_A(3,1) + p_B(2,1)\Delta u_B(2,2) = p_B v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = p_B$. Hence, $E(2,1) = p_B \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$, where $p_B = \frac{v_B^r}{\alpha v_A^r + v_B^r}$.

Likewise, at $(1,2)$, we have $\Delta u_A(1,2) = p_A(1,2)\Delta u_A(2,2) + p_B(1,2)\Delta u_B(1,3) = p_A v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = p_A$. Hence $E(1,2) = p_A \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$, where $p_A = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r}$.

In battle 3, at $(1,1)$, $\Delta u_A(1,1) = p_A(1,1)\Delta u_A(2,1) + p_B(1,1)\Delta u_A(1,2) = 2p_A p_B v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = 2p_A p_B$. Hence, $E(1,1) = 2p_A p_B \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

At $(2,0)$, $\Delta u_A(2,0) = p_A(2,0)\Delta u_A(3,0) + p_B(2,0)\Delta u_A(2,1) = p_B^2 v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = p_B^2$. Hence, $E(2,0) = p_B^2 \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

At $(0,2)$, $\Delta u_A(0,2) = p_A(0,2)\Delta u_A(1,2) + p_B(0,2)\Delta u_A(0,3) = p_A^2 v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = p_A^2$. Hence, $E(0,2) = p_A^2 \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

In battle 2, at $(1,0)$, $\Delta u_A(1,0) = p_A(1,0)\Delta u_A(2,0) + p_B(1,0)\Delta u_A(1,1) = 3p_A p_B^2 v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = 3p_A p_B^2$. Hence, $E(1,0) = 3p_A p_B^2 \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

At $(0,1)$, $\Delta u_A(0,1) = p_A(0,1)\Delta u_A(1,1) + p_B(0,1)\Delta u_A(0,2) = 3p_A^2 p_B v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = 3p_A^2 p_B$. Hence, $E(0,1) = 3p_A^2 p_B \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

In battle 1, at $(0,0)$, $\Delta u_A(0,0) = p_A(0,0)\Delta u_A(1,0) + p_B(0,0)\Delta u_A(0,1) = 6p_A^2 p_B^2 v_A$, which implies that $\frac{\Delta u_A(n_1,n_2)}{v_A} = 6p_A^2 p_B^2$. Hence, $E(0,0) = 6p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A+v_B)}{(\alpha v_A^r + v_B^r)^2}$.

Therefore, the expected aggregate effort equals

$$\begin{aligned} TE &= E(0,0) \\ &\quad + p_A E(1,0) + p_B E(0,1) \\ &\quad + p_A^2 E(2,0) + p_B^2 E(0,2) + 2p_A p_B E(1,1) \end{aligned}$$

$$
\begin{aligned}
&+3p_A^2 p_B E(2,1) + 3p_A p_B^2 E(1,2) \\
&+6p_A^2 p_B^2 E(2,2) \\
= \quad & 6p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} \\
&+3p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} + 3p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} \\
&+p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} + p_B^2 p_A^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} + 4p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} \\
&+3p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} + 3p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} \\
&+6p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2} \\
= \quad & 30p_A^2 p_B^2 \frac{\alpha r v_A^r v_B^r (v_A + v_B)}{(\alpha v_A^r + v_B^r)^2},
\end{aligned}
$$

where $p_A = \frac{\alpha v_A^r}{\alpha v_A^r + v_B^r}$ and $p_B = \frac{v_B^r}{\alpha v_A^r + v_B^r}$.

5.5 Example 2: 5-battle contest with outcome-dependent heterogeneity

We solve the game backwards. By Property 1, at state (n_1, n_2), the effort in current battle equals $E(n_1, n_2) = \beta(v_A, v_B, \alpha(n_1, n_2), r(n_1, n_2)) \Delta u(n_1, n_2)$. We apply the formula to compute the effort in each battle at each possible state as follows.

In battle 5, at $(2,2)$, $\Delta u_A(2,2) = v_A$ and thus $\Delta u(2,2) = \frac{\Delta u_A(n_1,n_2)}{v_A} = 1$. By Property 1, $E(2,2) = \beta(2,2) \Delta u(2,2) = \beta(2,2)$.

In battle 4, at $(2,1)$, $\Delta u_A(2,1) = p_A(3,1)\Delta u_A(3,1) + p_B(2,2)\Delta u_A(2,2) = p_B(2,2)\Delta u_A(2,2) = p_B(2,2)v_A$, which implies that $E(2,1) = \beta(2,1) \Delta u(2,1) = p_B(2,2)\beta(2,1)$, where $\Delta u(2,1) = p_B(2,2)$.

At $(1,2)$, $\Delta u_A(1,2) = p_A(2,2)\Delta u_A(2,2) + p_B(1,3)\Delta u_B(1,3) = p_A(2,2)\Delta u_A(2,2) = p_A(2,2)v_A$, which implies that $E(1,2) = \beta(1,2) \Delta u(1,2) = p_A(2,2)\beta(1,2)$, where $\Delta u(1,2) = p_A(2,2)$.

In battle 3, at $(1,1)$, $\Delta u_A(1,1) = p_A(2,1)\Delta u_A(2,1)+p_B(1,2)\Delta u_A(1,2) = p_A(2,1)p_B(2,2)v_A+ p_B(1,2)p_A(2,2)v_A$, which implies that $E(1,1) = \beta(1,1)\,\Delta u(1,1)$, where $\Delta u\,(1,1) = p_A(2,1)\Delta u(2,1)+ p_B(1,2)\Delta u(1,2) = p_A(2,1)p_B(2,2) + p_B(1,2)p_A(2,2)$.

At $(2,0)$, $\Delta u_A(2,0) = p_A(3,0)\Delta u_A(3,0) + p_B(2,1)\Delta u_A(2,1) = p_B(2,1)p_B(2,2)v_A$, which implies that $E(2,0) = \beta(2,0)\,\Delta u(2,0)$, where $\Delta u(2,0) = p_B(2,1)\Delta u(2,1) = p_B(2,1)p_B(2,2)$.

At $(0,2)$, $\Delta u_A(0,2) = p_A(1,2)\Delta u_A(1,2) + p_B(0,3)\Delta u_A(0,3) = p_A(1,2)\Delta u_A(1,2) = p_A(1,2)p_A(2,2)v_A$, which implies that $E(0,2) = \beta(0,2)\,\Delta u(0,2)$, where $\Delta u(0,2) = p_A(1,2)\Delta u(1,2) = p_A(1,2)p_A(2,2)$.

In battle 2, at $(1,0)$, player $A(2)$'s effective prize spread is

$$\begin{aligned} & \Delta u_A(1,0) \\ = \; & p_A(2,0)\Delta u_A(2,0) + p_B(1,1)\Delta u_A(1,1) \\ = \; & \left[\begin{array}{c} p_A(2,0)p_B(2,1)p_B(2,2) + p_B(1,1)p_A(2,1)p_B(2,2) \\ +p_B(1,1)p_B(1,2)p_A(2,2) \end{array} \right] v_A, \end{aligned}$$

which implies that

$$\begin{aligned} & \Delta u(1,0) \\ = \; & \frac{\Delta u_A(n_1,n_2)}{v_A} \\ = \; & p_A(2,0)p_B(2,1)p_B(2,2) + p_B(1,1)p_A(2,1)p_B(2,2) + p_B(1,1)p_B(1,2)p_A(2,2). \end{aligned}$$

Hence,

$$E(1,0) = \beta(1,0)\,\Delta u(1,0).$$

At $(0,1)$, player $A(2)$'s effective prize spread is

$$\begin{aligned} & \Delta u_A(0,1) \\ = \; & p_A(1,1)\Delta u_A(1,1) + p_B(0,2)\Delta u_A(0,2) \\ = \; & \left[\begin{array}{c} p_A(1,1)p_A(2,1)p_B(2,2) + p_A(1,1)p_B(1,2)p_A(2,2) \\ +p_B(0,2)p_A(1,2)p_A(2,2) \end{array} \right] v_A; \end{aligned}$$

which implies that

$$\begin{aligned}
&\Delta u(0,1) \\
= \; & \frac{\Delta u_A(n_1,n_2)}{v_A} \\
= \; & p_A(1,1)p_A(2,1)p_B(2,2) + p_A(1,1)p_B(1,2)p_A(2,2) + p_B(0,2)p_A(1,2)p_A(2,2).
\end{aligned}$$

Hence, the resulting effort is

$$E(0,1) = \beta\,(0,1)\,\Delta u(0,1).$$

In battle 1, at $(0,0)$, player $A(1)$'s effective prize spread is

$$\begin{aligned}
&\Delta u_A(0,0) \\
= \; & p_A(1,0)\Delta u_A(1,0) + p_B(0,1)\Delta u_A(0,1) \\
= \; & p_A(1,0)\left[p_A(2,0)p_B(2,1)p_B(2,2) + p_B(1,1)p_A(2,1)p_B(2,2) + p_B(1,1)p_B(1,2)p_A(2,2)\right] \\
& + p_B(0,1)\left[p_A(1,1)p_A(2,1)p_B(2,2) + p_A(1,1)p_B(1,2)p_A(2,2) + p_B(0,2)p_A(1,2)p_A(2,2)\right];
\end{aligned}$$

which implies that

$$\begin{aligned}
&\Delta u(0,0) \\
= \; & \frac{\Delta u_A(n_1,n_2)}{v_A} \\
= \; & p_A(1,0)p_A(2,0)p_B(2,1)p_B(2,2) + p_A(1,0)p_B(1,1)p_A(2,1)p_B(2,2) \\
& + p_A(1,0)p_B(1,1)p_B(1,2)p_A(2,2) + p_B(0,1)p_A(1,1)p_A(2,1)p_B(2,2) \\
& + p_B(0,1)p_A(1,1)p_B(1,2)p_A(2,2) + p_B(0,1)p_B(0,2)p_A(1,2)p_A(2,2).
\end{aligned}$$

One can also calculate $\Delta u_B(n_1,n_2)$ and easily verify that $\frac{\Delta u_A(n_1,n_2)}{v_A} = \frac{\Delta u_B(n_1,n_2)}{v_B}$.

Hence, the resulting effort is

$$TE(0,0) = \beta\,(0,0)\,\Delta u(0,0).$$

Therefore, the expected aggregate effort is

$$\begin{aligned} TE &= TE(0,0) \\ &+p_A(0,0)TE(1,0) + p_B(0,0)TE(0,1) \\ &+P(2,0)TE(2,0) + P(0,2)TE(0,2) + P(1,1)TE(1,1) \\ &+P(2,1)TE(2,1) + P(1,2)TE(1,2) \\ &+P(2,2)TE(2,2), \end{aligned}$$

where

$$\begin{aligned} TE(0,0) &= \begin{array}{c} p_A(1,0)p_A(2,0)p_B(2,1)p_B(2,2) + p_A(1,0)p_B(1,1)p_A(2,1)p_B(2,2) \\ +p_A(1,0)p_B(1,1)p_B(1,2)p_A(2,2) + p_B(0,1)p_A(1,1)p_A(2,1)p_B(2,2) \\ +p_B(0,1)p_A(1,1)p_B(1,2)p_A(2,2) + p_B(0,1)p_B(0,2)p_A(1,2)p_A(2,2) \end{array} \beta\,(0,0)\,; \\ TE(1,0) &= \left[\begin{array}{c} p_A(2,0)p_B(2,1)p_B(2,2) + p_B(1,1)p_A(2,1)p_B(2,2) \\ +p_B(1,1)p_B(1,2)p_A(2,2) \end{array} \right] \beta\,(1,0)\,; \\ TE(0,1) &= \left[\begin{array}{c} p_A(1,1)p_A(2,1)p_B(2,2) + p_A(1,1)p_B(1,2)p_A(2,2) \\ +p_B(0,2)p_A(1,2)p_A(2,2) \end{array} \right] \beta\,(0,1)\,; \\ TE(2,0) &= p_B(2,1)p_B(2,2)\beta\,(2,0)\,; \\ TE(0,2) &= p_A(1,2)p_A(2,2)\beta\,(0,2)\,; \\ TE(1,1) &= (p_A(2,1)p_B(2,2) + p_B(1,2)p_A(2,2))\,\beta\,(1,1)\,; \\ TE(2,1) &= p_B(2,2)\beta\,(2,1)\,; \\ TE(1,2) &= p_A(2,2)\beta\,(1,2)\,; \\ TE(2,2) &= \beta\,(2,2)\,, \end{aligned}$$

and $P(n_1, n_2)$ is the probability of reaching the state (n_1, n_2) from $(0,0)$. For example, $P(2,0) = p_A(0,0)p_A(1,0)$, $P(0,2) = p_B(0,0)p_B(0,1)$, and $P(1,1) = p_A(0,0)p_B(1,0) + p_B(0,0)p_A(0,1)$.

5.6 Proof of Lemma 2

By Definition 2, we prove the lemma by discussing three cases as follows.

Case 1: At (k, n_2) such that $n_2 \leq k-1$, by definition,

$$\begin{aligned}
\Delta u_A(k,n_2) &= V_A(k+1,n_2) - V_A(k,n_2+1) \\
&= v_A - [p_A(k,n_2+1)v_A + p_B(k,n_2+1)V_A(k,n_2+2)] \\
&= p_B(k,n_2+1)\left[v_A - V_A(k,n_2+2)\right].
\end{aligned}$$

In addition,

$$\begin{aligned}
& p_A(k+1,n_2)\Delta u_A(k+1,n_2) + p_B(k,n_2+1)\Delta u_A(k,n_2+1) \\
= \; & p_A(k+1,n_2)\cdot 0 + p_B(k,n_2+1)\left[V_A(k+1,n_2+1) - V_A(k,n_2+2)\right] \\
= \; & p_B(k,n_2+1)\left[v_A - V_A(k,n_2+2)\right].
\end{aligned}$$

Hence, $\Delta u_A(k,n_2) = p_A(k+1,n_2+1)\Delta u_A(k+1,n_2) + p_B(k,n_2+1)\Delta u_A(k,n_2+1)$. Analogously, $\Delta u_B(k,n_2) = p_A(k+1,n_2)\Delta u_B(k+1,n_2) + p_B(k,n_2+1)\Delta u_B(k,n_2+1)$.

Case 2: At (n_1, k) such that $n_1 \leq k-1$, by definition,

$$\begin{aligned}
\Delta u_A(n_1,k) &= V_A(n_1+1,k) - V_A(n_1,k+1) \\
&= [p_A(n_1+1,k)V_A(n_1+2,k) + p_B(n_1+1,k)V_A(n_1+1,k+1)] - v_A \\
&= p_A(n_1+1,k)\left[V_A(n_1+2,k) - v_A\right].
\end{aligned}$$

In addition,

$$p_A(n_1+1,k)\Delta u_A(n_1+1,k) + p_B(n_1,k+1)\Delta u_A(n_1,k+1)$$

$$
\begin{aligned}
&= p_A(n_1+1,k)\left[V_A(n_1+2,k) - V_A(n_1+1,k+1)\right] + p_B \cdot 0 \\
&= p_A(n_1+1,k)\left[V_A(n_1+2,k) - v_A\right].
\end{aligned}
$$

Hence, $\Delta u_A(n_1,k) = p_A(n_1+1,k)\Delta u_A(n_1+1,k) + p_B(n_1,k+1)\Delta u_A(n_1,k+1)$. Analogously, $\Delta u_B(n_1,k) = p_A(n_1+1,k)\Delta u_B(n_1+1,k) + p_B(n_1,k+1)\Delta u_B(n_1,k+1)$.

Case 3: At (n_1,n_2) such that $n_1,n_2 \leq k-1$,

$$
\begin{aligned}
\Delta u_A(n_1,n_2) &= V_A(n_1+1,n_2) - V_A(n_1,n_2+1) \\
&= \begin{array}{c} p_A(n_1+1,n_2)V_A(n_1+2,n_2) + p_B(n_1+1,n_2)V_A(n_1+1,n_2+1) \\ -p_A(n_1,n_2+1)V_A(n_1+1,n_2+1) - p_B(n_1,n_2+1)V_A(n_1,n_2+2) \end{array} \\
&= \begin{array}{c} V_A(n_1+1,n_2+1) + p_A(n_1+1,n_2)\left(V_A(n_1+2,n_2) - V_A(n_1+1,n_2+1)\right) \\ -V_A(n_1,n_2+2) - p_A(n_1,n_2+1)\left(V_A(n_1+1,n_2+1) - V_A(n_1,n_2+2)\right) \end{array} \\
&= \Delta u_A(n_1,n_2+1) + p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) - p_A(n_1,n_2+1)\Delta u_A(n_1,n_2+1) \\
&= p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1).
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
\Delta u_B(n_1,n_2) &= V_B(n_1,n_2+1) - V_B(n_1+1,n_2) \\
&= \begin{array}{c} p_A(n_1,n_2+1)V_B(n_1+1,n_2+1) + p_B(n_1,n_2+1)V_B(n_1,n_2+2) \\ -p_A(n_1+1,n_2)V_B(n_1+2,n_2) - p_B(n_1+1,n_2)V_B(n_1+1,n_2+1) \end{array} \\
&= \begin{array}{c} V_B(n_1+1,n_2+1) + p_B(n_1,n_2+1)\left(V_B(n_1,n_2+2) - V_B(n_1+1,n_2+1)\right) \\ -V_B(n_1+2,n_2) - p_B(n_1+1,n_2)\left(V_B(n_1+1,n_2+1) - V_B(n_1+2,n_2)\right) \end{array} \\
&= \Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1) - p_B(n_1+1,n_2)\Delta u_B(n_1+1,n_2) \\
&= p_A(n_1+1,n_2)\Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1).
\end{aligned}
$$

In sum, $\Delta u_A(n_1,n_2) = p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1)$ and $\Delta u_B(n_1,n_2) = p_A(n_1+1,n_2)\Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1)$.

5.7 Proof of Lemma 3

At (k,k), $\Delta u_A(k,k) = v_A$ and $\Delta u_B(k,k) = v_B$. The formula holds automatically. We prove the lemma by mathematical induction.

Case 1: To show the lemma hold for (k, n_2) with $n_2 \leq k-1$, suppose the lemma holds for $n_2 \geq t+1$, where $t \leq k-1$, we want to prove that it holds for $n_2 \geq t$, which is equivalent to proving the formula holds at (k,t).

At (k,t), it follows from Lemma 2 that

$$\begin{aligned}\Delta u_A(k,t) &= p_A(k+1,t)\Delta u_A(k+1,t) + p_B(k,t+1)\Delta u_A(k,t+1) = p_B(k,t+1)\Delta u_A(k,t+1);\\ \Delta u_B(k,t) &= p_A u_B(k+1,t) + p_B \Delta u_B(k,t+1) = p_B \Delta u_B(k,t+1).\end{aligned}$$

Since $\Delta u_A(k,t+1)/v_A = \Delta u_B(k,t+1)/v_B$, $\Delta u_A(k,t)/v_A = \Delta u_B(k,t)/v_B$ holds.

Case 2: For (k, n_2) with $n_2 \leq k-1$, the proof is analogous to Case 1.

Case 3: To establish the lemma for (n_1, n_2) such that $n_1, n_2 \leq k-1$, suppose the lemma holds for (n_1+1, n_2) and (n_1, n_2+1), it suffices to show that the lemma holds for (n_1, n_2), which is true, since

$$\begin{aligned}\Delta u_A(n_1,n_2) &= p_A(n_1+1,n_2)\Delta u_A(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_A(n_1,n_2+1);\\ \Delta u_B(n_1,n_2) &= p_A(n_1+1,n_2)\Delta u_B(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u_B(n_1,n_2+1).\end{aligned}$$

using Lemma 2.

5.8 Proof of Proposition 1

Recall that it follows from (5) and Property 1 that $E(n_1,n_2) = \beta(v_A, v_B, \alpha, r)\Delta u(n_1,n_2)$. We therefore rewrite (7) as

$$\begin{aligned}\mathbf{TE}_{\text{Homo}} &= \sum_{(n_1,n_2)\in S/E} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} E(n_1,n_2)\\ &= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} \beta(v_A, v_B, \alpha, r)\Delta u(n_1,n_2).\end{aligned}$$

Step 1: We prove $\Delta u(n_1,n_2)=\gamma(n_1,n_2)p^{k-n_1}\left(1-p\right)^{k-n_2}$ by induction, where $\gamma(n_1,n_2)$ is a coefficient that solely relies on (n_1,n_2). We will determine $\gamma(n_1,n_2)$ in Step 2.

First, it follows from direct calculation that $\Delta u(n_1,n_2)=\Delta u_A(k,k)/v_A=1=p^{k-k}\left(1-p\right)^{k-k}$. Suppose now that the result holds at state (n_1+1,n_2) and (n_1,n_2+1), i.e., $\Delta u(n_1+1,n_2)=\gamma(n_1+1,n_2)p^{k-n_1-1}\left(1-p\right)^{k-n_2}$ and $\Delta u(n_1,n_2+1)=\gamma(n_1,n_2+1)p^{k-n_1}\left(1-p\right)^{k-n_2-1}$, it remains to show that $\Delta u(n_1,n_2)=\gamma(n_1,n_2)p^{k-n_1}\left(1-p\right)^{k-n_2}$, which holds since

$$\begin{aligned}
& \Delta u(n_1,n_2) \\
= \;& p\Delta u(n_1+1,n_2)+(1-p)\Delta u(n_1,n_2+1) \\
= \;& \gamma(n_1+1,n_2)p^{k-n_1}\left(1-p\right)^{k-n_2}+\gamma(n_1,n_2+1)p^{k-n_1}\left(1-p\right)^{k-n_2} \\
= \;& \left[\gamma(n_1+1,n_2)+\gamma(n_1,n_2+1)\right]p^{k-n_1}\left(1-p\right)^{k-n_2} \\
= \;& \gamma(n_1,n_2)p^{k-n_1}\left(1-p\right)^{k-n_2},
\end{aligned}$$

where $\gamma(n_1,n_2):=\gamma(n_1+1,n_2)+\gamma(n_1,n_2+1)$.

Step 2: For any $t\in\{1,2,...,2k\}$, we show that $\sum_{n_1+n_2=t}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2)=C_{2k}^{k}$ holds by induction.

First, it holds at (k,k) clearly, as $C_{2k}^{k}\gamma(k,k)=C_{2k}^{k}$. Suppose $\sum_{(n_1+n_2)=t+1}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2)=C_{2k}^{k}$ holds, we want to prove $\sum_{(n_1+n_2)=t}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2)=C_{2k}^{k}$. For that, we consider

$$\begin{aligned}
& \sum_{(n_1+n_2)=t}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2) \\
= \;& \sum_{(n_1+n_2)=t}C_{n_1+n_2}^{n_1}\left[\gamma(n_1+1,n_2)+\gamma(n_1,n_2+1)\right] \\
= \;& \sum_{(n_1+n_2)=t+1}C_{n_1+n_2-1}^{n_1-1}\gamma(n_1,n_2)+\sum_{(n_1+n_2)=t+1}C_{n_1+n_2-1}^{n_1}\gamma(n_1,n_2) \\
= \;& \sum_{(n_1+n_2)=t+1}\frac{n_1}{n_1+n_2}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2)+\sum_{(n_1+n_2)=t+1}\frac{n_2}{n_1+n_2}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2) \\
= \;& \sum_{(n_1+n_2)=t+1}C_{n_1+n_2}^{n_1}\gamma(n_1,n_2) \\
= \;& C_{2k}^{k}.
\end{aligned}$$

With the results in Steps 1 and 2,

$$\begin{aligned}
\mathbf{TE}_{\text{Homo}} &= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} p^{n_1} (1-p)^{n_2} \beta(v_A, v_B, \alpha, r) \Delta u(n_1, n_2) \\
&= \sum_{(n_1,n_2)} C_{n_1+n_2}^{n_1} \gamma(n_1, n_2) \beta(v_A, v_B, \alpha, r) p^k (1-p)^k \\
&= \sum_{t=1}^{2k} \sum_{n_1+n_2=t} C_{n_1+n_2}^{n_1} \gamma(n_1, n_2) \beta(v_A, v_B, \alpha, r) p^k (1-p)^k \\
&= \sum_{t=1}^{2k} C_{2k}^{k} \beta(v_A, v_B, \alpha, r) p^k (1-p)^k \\
&= (2k+1) C_{2k}^{k} p^k (1-p)^k \beta(v_A, v_B, \alpha, r).
\end{aligned}$$

5.9 Proof of Proposition 2

By Property 1, the expected total effort equals

$$\begin{aligned}
\mathbf{TE}_{Hete,k} &= \sum_{(n_1,n_2)\in S/E} P(n_1, n_2) E(n_1, n_2) \\
&= \sum_{(n_1,n_2)\in S/E} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \beta(n_1, n_2) \Delta u(n_1, n_2),
\end{aligned}$$

where $P(n_1, n_2) = \sum_{g\in G((0,0),(n_1,n_2))} P(g)$ is the probability that the state (n_1, n_2) is reached.

It then remains to show that

$$\Delta u(n_1, n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g).$$

We prove the above equation by induction. For $(n_1, n_2) = (k-1, k)$, on one hand, it follows from (11) that $\Delta u(k-1, k) = \widehat{P}((k-1, k), (k, k)) = p_A(k, k)$. On the other hand, by direct calculation or (3), we have

$$\Delta u(k-1, k) = \Delta u_A(k-1, k)/v_A$$

$$
\begin{aligned}
&= \left(u_A(k,k) - u_A(k-1,k+1)\right)/v_A \\
&= \left(p_A(k,k)u_A(k+1,k) + p_B(k,k)u_A(k,k+1)\right)/v_A \\
&= p_A(k,k),
\end{aligned}
$$

which means (11) holds for state $(k-1,k)$.

Analogously, for $(n_1,n_2) = (k,k-1)$, by direct calculation or (3), we have

$$
\begin{aligned}
\Delta u(k,k-1) &= \Delta u_A(k,k-1)/v_A \\
&= \left(u_A(k+1,k-1) - u_A(k,k)\right)/v_A \\
&= \left(v_A - p_A(k,k)u_A(k+1,k) + p_B(k,k)u_A(k,k+1)\right)/v_A \\
&= p_B(k,k),
\end{aligned}
$$

which coincides with the expression derived from (11).

Suppose that the formula holds for (n_1+1,n_2) and (n_1,n_2+1), i.e., $\Delta u(n_1+1,n_2) = \sum_{g\in G((n_1+1,n_2),(k,k))} \widehat{P}(g)$ and $\Delta u(n_1,n_2+1) = \sum_{g\in G((n_1,n_2+1),(k,k))} \widehat{P}(g)$ hold. By applying (6), we have

$$
\begin{aligned}
&\Delta u(n_1,n_2) \\
= \;& p_A(n_1+1,n_2)\Delta u(n_1+1,n_2) + p_B(n_1,n_2+1)\Delta u(n_1,n_2+1) \\
= \;& p_A(n_1+1,n_2) \sum_{g\in G((n_1+1,n_2),(k,k))} \widehat{P}(g) + p_B(n_1,n_2+1) \sum_{g\in G((n_1,n_2+1),(k,k))} \widehat{P}(g) \\
= \;& \widehat{p}((n_1,n_2),(n_1+1,n_2)) \sum_{g\in G((n_1+1,n_2),(k,k))} \widehat{P}(g) + \widehat{p}((n_1,n_2),(n_1,n_2+1)) \sum_{g\in G((n_1,n_2+1),(k,k))} \widehat{P}(g) \\
= \;& \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g),
\end{aligned}
$$

using Definition 3. In other words, (11) holds for any $(n_1,n_2) \in S/E$.

5.10 Proof of Corollary 1

$$\begin{aligned}
\mathbf{TE}_{Hete,k} &= \sum_{(n_1,n_2)\in S/E} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2) \\
&= \sum_{t\in\{1,...,2k+1\}} \sum_{n_1+n_2=t} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta.
\end{aligned}$$

When α and r are invariant across the states, $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) = \sum_{g\in G((n_1,n_2),(k,k))} P(g)$, since the winning probability $p := p_A(n_1,n_2)$ does not depend on the state. As a result, for a state $(n_1,n_2) \in S/E$, $\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)$ equals the probability of a particular path that connects $(0,0)$ and (k,k) and crosses state (n_1,n_2). Therefore, for any given $t \in \{1,...,2k+1\}$, $\sum_{n_1+n_2=t} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)$ equals the probability of a path connecting $(0,0)$ and (k,k), which implies that

$$\begin{aligned}
& \sum_{n_1+n_2=t} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \\
=& \sum_{n_1+n_2=t} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} P(g) \right) \\
=& \sum_{g\in G((0,0),(k,k))} P(g) \\
=& C_{2k}^{k} p^k (1-p)^k.
\end{aligned}$$

By direct substitution, we have

$$\begin{aligned}
\mathbf{TE}_{Hete,k} &= \sum_{t\in\{1,...,2k+1\}} C_{2k}^{k} p^k (1-p)^k \beta \\
&= (2k+1) C_{2k}^{k} p^k (1-p)^k \beta. \\
&= \mathbf{TE}_{Homo,k}.
\end{aligned}$$

5.11 Proof of Lemma 5

The total effort function in (10) can be rewritten as

$$\begin{aligned}
\mathbf{TE}_{Hete,k} &= \sum_{(n_1,n_2)\in S/E}\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)\beta(n_1,n_2) \\
&= \sum_{t\in\{0,\ldots,2k\}}\sum_{n_1+n_2=t}\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)\beta(n_1,n_2).
\end{aligned}$$

Depending on the state (n_1,n_2), $p_A(n_1',n_2')$ may affect the probability of a path and players' incentive along a path, i.e., $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ and $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$. Recall that, by 3, $p((n_1',n_2'),(n_1'+1,n_2')) = p_A(n_1',n_2')$, $p((n_1',n_2'),(n_1',n_2'+1)) = p_B(n_1',n_2')$, $\widehat{p}((n_1'-1,n_2'),(n_1',n_2')) = p_A(n_1',n_2')$, and $\widehat{p}((n_1',n_2'-1),(n_1',n_2')) = p_B(n_1',n_2')$.

To proceed, we first rewrite the expected aggregate effort as

$$\begin{aligned}
&\mathbf{TE}_{Hete,k} \\
&= \sum_{t\in\{0,\ldots,2k\}}\sum_{\substack{n_1+n_2=t\\(n_1,n_2)\in S/E}}\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)\beta(n_1,n_2) \\
&= \sum_{t\in\{0,\ldots,t'\}}\sum_{\substack{n_1+n_2=t\\(n_1,n_2)\in S/E}}\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)\beta(n_1,n_2) \\
&\quad + \sum_{t\in\{t'+1,\ldots,2k\}}\sum_{\substack{n_1+n_2=t\\(n_1,n_2)\in S/E}}\left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right)\left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right)\beta(n_1,n_2),
\end{aligned}$$

where

$$\sum_{g\in G((0,0),(n_1,n_2))} P(g)$$

$$
\begin{aligned}
= & \begin{array}{c} \left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\right)\left(\sum_{g\in G((n_1',n_2'),(n_1,n_2))} P(g)\right) \\ +\sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \left(\sum_{g\in G((0,0),(n_1'',n_2''))} P(g)\right)\left(\sum_{g\in G((n_1'',n_2''),(n_1,n_2))} P(g)\right) \end{array} \\
= & \begin{array}{c} \left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\right)\left(\begin{array}{c} p_A(n_1',n_2')\sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) \\ +(1-p_A(n_1',n_2'))\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \end{array}\right) \\ +\sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \left(\sum_{g\in G((0,0),(n_1'',n_2''))} P(g)\right)\left(\sum_{g\in G((n_1'',n_2''),(n_1,n_2))} P(g)\right) \end{array},
\end{aligned}
$$

which means that $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ is a linear function of $p_A(n_1',n_2')$ for any (n_1,n_2) and the coefficient of $p_A(n_1',n_2')$ in $\sum_{g\in G((0,0),(n_1,n_2))} P(g)$ equals

$$
\left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\right)\left(\sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g)-\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g)\right).
$$

Since $p_A(n_1',n_2')$ would only affect the probabilities of the paths after battle $t'=n_1'+n_2'+1$, its effect on the expected aggregate effort through affecting the probability of paths can be summarized by

$$
\begin{aligned}
I = & \sum_{t\in\{t'+1,\ldots,2k\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left[\left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\right)\left(\begin{array}{c} \sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) \\ -\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \end{array}\right)\right] \\
& \cdot\left[\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)\right]\beta(n_1,n_2) \\
= & \sum_{t\in\{t'+1,\ldots,2k\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1',n_2'))} P(g)\right)\left(\begin{array}{c} \sum_{g\in G((n_1'+1,n_2'),(n_1,n_2))} P(g) \\ -\sum_{g\in G((n_1',n_2'+1),(n_1,n_2))} P(g) \end{array}\right) \\
& \cdot\Delta u(n_1,n_2)\beta(n_1,n_2),
\end{aligned}
$$

using $\Delta u(n_1,n_2)=\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$.

Analogously, to determine how $p_A(n_1',n_2')$ affects the incentives, i.e., $\widehat{P}(g)$, we rewrite

$$
\mathbf{TE}_{Hete,k}
$$

$$= \sum_{t\in\{0,\dots,2k\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2)$$

$$= \sum_{t\in\{0,\dots,t'-1\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2)$$

$$+ \sum_{t\in\{t',\dots,2k\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g) \right) \left(\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g) \right) \beta(n_1,n_2),$$

where

$$\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$$

$$= \begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1',n_2'))} \widehat{P}(g) \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \sum_{g\in G((n_1,n_2),(n_1'',n_2''))} \widehat{P}(g) \left(\sum_{g\in G((n_1'',n_2''),(k,k))} \widehat{P}(g) \right) \end{array}$$

$$= \begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g)\widehat{P}((n_1'-1,n_2'),(n_1',n_2')) \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g)\widehat{P}((n_1',n_2'-1),(n_1',n_2')) \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \sum_{g\in G((n_1,n_2),(n_1'',n_2''))} \widehat{P}(g) \left(\sum_{g\in G((n_1'',n_2''),(k,k))} \widehat{P}(g) \right) \end{array}$$

$$= \begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) p_B(n_1',n_2') \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) p_A(n_1',n_2') \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \sum_{g\in G((n_1,n_2),(n_1'',n_2''))} \widehat{P}(g) \left(\sum_{g\in G((n_1'',n_2''),(k,k))} \widehat{P}(g) \right) \end{array}$$

$$= \begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) \left(1 - p_A(n_1',n_2')\right) \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) p_A(n_1',n_2') \left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g) \right) \\ + \sum_{\substack{n_1''+n_2''=t' \\ (n_1'',n_2'')\neq(n_1',n_2')}} \sum_{g\in G((n_1,n_2),(n_1'',n_2''))} \widehat{P}(g) \left(\sum_{g\in G((n_1'',n_2''),(k,k))} \widehat{P}(g) \right) \end{array}.$$

which means that $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ is a linear function of $p_A(n_1',n_2')$ for any (n_1,n_2) and the coefficient of $p_A(n_1', n_2')$ in $\sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$ equals

$$\left(-\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) + \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g)\right)\left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g)\right).$$

Since $p_A(n_1', n_2')$ would only affect the incentives, i.e., $\widehat{P}(g)$ before battle $t'-1 = n_1' + n_2'$, its effect on the expected aggregate effort through affecting the players' incentive along the paths can be summarized by

$$\begin{aligned} II &= \sum_{t\in\{0,\ldots,t'-1\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left[\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right] \\ &\quad \cdot \left[\left(\begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) \\ -\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) \end{array}\right)\left(\sum_{g\in G((n_1',n_2'),(k,k))} \widehat{P}(g)\right)\right]\beta(n_1,n_2) \\ &= \sum_{t\in\{0,\ldots,t'-1\}} \sum_{\substack{n_1+n_2=t \\ (n_1,n_2)\in S/E}} \left(\sum_{g\in G((0,0),(n_1,n_2))} P(g)\right) \\ &\quad \cdot \left(\begin{array}{c} \sum_{g\in G((n_1,n_2),(n_1'-1,n_2'))} \widehat{P}(g) \\ -\sum_{g\in G((n_1,n_2),(n_1',n_2'-1))} \widehat{P}(g) \end{array}\right)\Delta u(n_1',n_2')\beta(n_1,n_2), \end{aligned}$$

using $\Delta u(n_1, n_2) = \sum_{g\in G((n_1,n_2),(k,k))} \widehat{P}(g)$.

Combining the results above, we conclude that the coefficient of $p_A(n_1', n_2')$ in $\mathbf{TE}_{Hete,k}$ is $I + II$.

5.12 Example 1 (Continue)

Consider a best-of-three team contest with outcome-dependent heterogeneity. It follows from Lemma 5 that the coefficient of $p_A(n_1', n_2')$ in $\mathbf{TE}_{\text{Hete}}$ is $\varphi(\boldsymbol{\alpha}/\alpha(n_1', n_2'))$. To apply the formula to compute $\varphi(\boldsymbol{\alpha}/\alpha(0,0))$, let $(n_1', n_2') = (0,0)$, we have

$$I = P((0,0),(0,0))\left[P((1,0),(1,0)) - P((0,1),(1,0))\right]\widehat{P}((1,0),(1,1))\beta(1,0)$$

$$
\begin{aligned}
& +P((0,0),(0,0))\left[P((1,0),(0,1))-P((0,1),(0,1))\right]\widehat{P}((0,1),(1,1))\beta(0,1) \\
& +P((0,0),(0,0))\left[P((1,0),(1,1))-P((0,1),(1,1))\right]\widehat{P}((1,1),(1,1))\beta(1,1) \\
= \; & 1\cdot[1-0]\,p_B(1,1)\beta(1,0)+1\cdot[0-1]\,p_A(1,1)\beta(0,1) \\
& +1\cdot[p_B(1,0)-p_A(0,1)]\cdot 1\cdot\beta(1,1) \\
= \; & p_B(1,1)\beta(1,0)-p_A(1,1)\beta(0,1)+\left(p_B(1,0)-p_A(0,1)\right)\beta(1,1),
\end{aligned}
$$

and $II=0$. Hence, the coefficient of $p_A(0,0)$ in $TE_{Hete,k=1}$ is

$$
\varphi(\boldsymbol{\alpha}/\alpha(0,0))=I+II=p_B(1,1)\beta(1,0)-p_A(1,1)\beta(0,1)+\left(p_B(1,0)-p_A(0,1)\right)\beta(1,1).
$$

Recall that $TE_{Hete,k=1}$ can be decomposed into $\varphi(\boldsymbol{\alpha}/\alpha(0,0))p_A(0,0)+\omega(\boldsymbol{\alpha}/\alpha(0,0))\beta\,(0,0)$ and other terms that are independent of $\alpha(0,0)$, where $\omega(\boldsymbol{\alpha}/\alpha(0,0))=\sum_{g\in G((0,0),(1,1))}\widehat{P}(g)=p_A(1,0)p_B(1,1)+p_B(0,1)p_A(1,1)$.

In Example 2, we show that

$$
\begin{aligned}
TE_{Hete,k=1}(\boldsymbol{\alpha},\mathbf{r}) \;=\; & \left[p_A(1,0)p_B(1,1)+p_B(0,1)p_A(1,1)\right]\beta\,(0,0) \\
& +\mathbf{p}_A(\mathbf{0},\mathbf{0})p_B(1,1)\beta\,(1,0)+\mathbf{p}_B(\mathbf{0},\mathbf{0})p_A(1,1)\beta\,(0,1) \\
& +[\mathbf{p}_A(\mathbf{0},\mathbf{0})p_B(1,0)+\mathbf{p}_B(\mathbf{0},\mathbf{0})p_A(0,1)]\beta\,(1,1)\,.
\end{aligned}
$$

By rearrangement, we have

$$
\begin{aligned}
TE_{Hete,k=1}(\boldsymbol{\alpha},\mathbf{r}) \;=\; & p_A(0,0)\begin{bmatrix} p_B(1,1)\beta\,(1,0)-p_A(1,1)\beta\,(0,1) \\ +\left(p_B(1,0)-p_A(0,1)\right)\beta\,(1,1)\end{bmatrix} \\
& +p_A(1,1)\beta\,(0,1)+p_A(0,1)]\beta\,(1,1) \\
& +\left[p_A(1,0)p_B(1,1)+p_B(0,1)p_A(1,1)\right]\beta\,(0,0)\,,
\end{aligned}
$$

using $p_B(0,0)=1-p_A(0,0)$.

By direct comparison, the two approaches yield the same $\varphi(\boldsymbol{\alpha}/\alpha(0,0))$. Analogously, one can check that

$$\begin{aligned}
\varphi(\boldsymbol{\alpha}/\alpha(1,0)) &= p_B(1,1)\beta\,(0,0) - p_A(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(0,1)) &= -p_A(1,1)\beta\,(0,0) + p_B(0,0)\beta\,(1,1)\,; \\
\varphi(\boldsymbol{\alpha}/\alpha(1,1)) &= \left[-p_A(1,0) + p_B(0,1)\right]\beta\,(0,0) - p_A(0,0)\beta\,(1,0) + p_B(0,0)\beta\,(0,1)\,.
\end{aligned}$$

using the two approaches.

5.13 Example 2 (Continue)

For a best-of-five team contest with outcome-dependent heterogeneity, we consider two approaches to derive $\varphi(\boldsymbol{\alpha}/\alpha(1,2))$. The first approach relies on Lemma 5. The second one relies on the direct calculation using the expected aggregate effort derived in Example 2.

In the first approach, we apply the formula in Lemma 5 to compute $\varphi(\boldsymbol{\alpha}/\alpha(1,2))$. Let $(n_1', n_2') = (1,2)$, direct substitution yields

$$\begin{aligned}
I &= P((0,0),(1,2))\left[P((2,2),(2,2)) - P((1,3),(2,2))\right]\widehat{P}((2,2),(2,2))\beta(2,2) \\
&= P(1,2)\beta(2,2).
\end{aligned}$$

and

$$\begin{aligned}
II &= \left(p_B(0,1)p_B(0,2) - p_A(1,0)p_B(1,1) - p_B(0,1)p_A(1,1)\right)p_A(2,2)\beta(0,0) \\
&\quad -P(1,0)p_B(1,1)p_A(2,2)\beta(1,0) \\
&\quad +P(0,1)\left(p_B(0,2) - p_A(1,1)\right)p_A(2,2)\beta(0,1) \\
&\quad +P(0,2)p_A(2,2)\beta(0,2) - P(1,1)p_A(2,2)\beta(1,1),
\end{aligned}$$

using $P(n_1,n_2) := P((0,0),(n_1,n_2))$. By Lemma 5, $\varphi(\boldsymbol{\alpha}/\alpha(1,2)) = I + II$. Moreover, $TE_{Hete,k=2}(\boldsymbol{\alpha},\mathbf{r}) = \varphi(\boldsymbol{\alpha}/\alpha(1,2))p_A(1,2) + \omega(\boldsymbol{\alpha}/\alpha(1,2))\beta(1,2)$, where $\omega(\boldsymbol{\alpha}/\alpha(1,2)) := P(1,2)p_A(2,2)$.

For the second approach, we rewrite the expected aggregate effort in Example 2 as

$$TE_{Hete,k=2}(\boldsymbol{\alpha},\mathbf{r})$$

$$= \left[\begin{array}{c} \left(-p_A(1,0)p_B(1,1) - p_B(0,1)p_A(1,1) + p_B(0,1)p_B(0,2)\right) p_A(2,2)\beta(0,0) \\ -p_A(0,0)p_B(1,1)p_A(2,2)\beta\left(1,0\right) \\ +p_B(0,0)\left(-p_A(1,1) + p_B(0,2)\right) p_A(2,2)\beta\left(0,1\right) \\ +P(0,2)p_A(2,2)\beta\left(0,2\right) - P(1,1)p_A(2,2)\beta\left(1,1\right) \\ +P(1,2)\beta\left(2,2\right) \end{array} \right] p_A(1,2)$$

$$+P(1,2)p_A(2,2)\beta\left(1,2\right)$$

$$+T(\boldsymbol{\alpha}/\alpha(1,2)),$$

using $p_B(1,2) = 1 - p_A(2,2)$ and $P(2,2) = P(1,2)p_A(1,2) + P(2,1)p_B(2,1)$, and $T(\boldsymbol{\alpha}/\alpha(1,2))$ denotes terms that do not involve $\alpha(1,2)$. By direct comparison, the two approaches generate the same result.

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