How to Split the Pie: Optimal Rewards in Dynamic Multi-Battle Competitions*

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Abstract

Multi-battle competitions are ubiquitous in real life. In this paper, we examine the effort-maximizing reward design in sequentially played multi-battle competitions. The organizer has a fixed prize budget, and rewards players contingent on the number of battles they win in a three-battle contest. Battles are played between two opposing players or between selected pairs of players from two opposing teams. A full spectrum of contest technologies in the Tullock family is accommodated. We find that the optimal design is implemented by a contest prize for the grand winner who wins the majority of battles together with uniform battle prizes to battle winners. For competitions between two individuals, the optimal design varies with the discriminatory power of the contest technology. When it is in the low range, winner-take-all is optimal. For the intermediate range, as discriminatory power increases, the optimal prize structure evolves continuously from winner-take-all to a proportional-division rule due to the need to mitigate the growing momentum/discouragement effect. For the high range, a whole span of prize structures extracts full surplus and thus is optimal. In contrast, winner-take-all is optimal for team competitions, regardless of the contest technology, in which the momentum/discouragement effect does not exist.

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1 Introduction

Dynamic multi-battle contests are abundant in reality. Many economic and social competitions, including research and development races, lawsuits/litigation, bidding for procurement contracts,

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policy debates, legislative, lobbying, electoral campaigns, and sports, can be viewed as contests in which opposing parties expend non-refundable, costly effort to compete in multiple battles.[1] A unique feature naturally arises from the multi-round nature of such competitions: The final winner of the overall contest, as well as contestants' rewards, is in general determined by the outcomes of *all* battles instead of a *single* battle. For example, in a widely adopted winner-take-all best-of-$(2n+1)$ contest, a party wins the contest if and only if the party wins the majority of the $2n+1$ battles, and the winner takes the entire prize.

There exists a wide range of diversity in the prize structures of multi-battle contests. Often, the final winner is determined according to the above-mentioned "majority" rule, and the prize is split between the winner and loser by fixed shares, which are not related to the number of battles won by the loser. This prize-allocation rule is widely observed in sports. In the finals of the 2013 US Open tennis tournament, for example, the winners received \$2.6 million, while the losers received \$1.3 million.[2] More saliently, two-party political campaign competitions (e.g. those to gain control of the legislature) have long been viewed as winner-take-all multi-battle contests (e.g., Snyder, 1989, Fu, Lu and Pan, 2015, and Boyer and Konrad, 2015), as well as the Democratic and Republican primaries that nominate the parties' candidates for the U.S. presidential election (e.g., Klumpp and Polborn, 2006).

On the other hand, in many instances, intermediate prizes are awarded to winners of component battles, and prize allocation can be contingent on the number of component battles each player wins. This type of prize structure is observed in labor tournaments. In many academic departments, faculty members are typically evaluated in three areas: teaching, research, and service. Excellent in one or two areas usually results in a performance bonus and/or pay increase, while excellent overall typically leads to extra and more significant benefits, including higher likelihood of promotion. Similar arrangements are prevalent in sports. The Fédération Internationale de Volleyball (FIVB) World League (for men) and World Grand Prix (for women) are examples of intermediate prizes for each single match. In the group stage of the tournament, each match is a best-of-five game; since 2010, the winning team earns 3 match points and the losing team receives 0 if the final set score is either 3-0 or 3-1. The winning team earns 2 match points and the losing team wins 1 match point if the final set score is 3-2.[3] In Formula I car races, each Grand Prix allots points to teams according to their ranks, and the grand championship as well as F1's prize money is awarded on an annual basis at the end of the season based on teams' championship points. A single point (a race win is worth 25 points) is, potentially, worth millions of dollars.[4]

[1] Refer to Konrad and Kovenock (2009), Konrad (2009), and Kovenock and Roberson (2012), among many others, for examples of multi-battle contests.

[2] Refer to http://www.examiner.com/article/prize-money-at-2013-u-s-open-tennis, accessed on May 28th, 2015. The prize for the runner-ups is permission to proceed to the finals. Therefore, the effective prize structure is a winner-take-all for the grand winners. Similarly, in the finals of the 2015 OUE Singapore Open, the winners received about \$23,000 and the loser received \$11,400. Refer to https://www.youtube.com/watch?v=tDgVoHm7MW4, accessed on May 28th, 2015.

[3] We thank Xiandeng Jiang for this example. Please refer to Jiang (2014) for details.

[4] Refer to http://www.grandprixevents.com/news/103/f1+teams+-+where+do+they+get+their+money+from%3F; http://www.f1fanatic.co.uk/2014/11/17/design-points-system-formula-one/; http://www.tsmplug.com/f1/formula-

More broadly, similar incentive structures prevail in multi-round lawsuits, competitions for procurement contracts as well as parliamentary debates on government policies, proposed legislation, and current issues. Different parties collect and provide evidence in multiple rounds to support their own interests/views. Eventually, the chance that a particular conclusion/outcome will be reached is determined by information revealed/collected and the outcomes of all rounds of hearings/debates. In this kind of examples, in order to make a convincing decision, more evidence and efforts in collecting evidence are preferred. And the chance of continuing the debates can be viewed as rewards, since that may affect the final outcome. The prize allocation rule essentially answers the question whether the debates should be terminated and a decision should be made accordingly when a party wins majority rounds. These practices demonstrate how prize allocations can be contingent on how many battles each party wins. In general, prize structures could depend on both battle outcomes and the overall contest outcome, but are not necessarily restricted to contest prizes and battle prizes.[5]

Interesting questions thus arise: How does contest organizers' choice of prize-allocation rules depend on the contest structure? In particular, for which situations should the allocation rule solely rely on the performance aggregated over all battles, i.e., the final winning status of the whole contest, and for which situations should the players be awarded separately in each individual battle, but not on their aggregate performance? Why do commonly observed prize structures usually take a simple form, such as the combination of a grand contest prize and component battle prizes, when contest organizers have access to more sophisticated contingent prize-allocation rules?

In this paper, we aim to rationalize commonly adopted prize-allocation rules from the perspective of effort elicitation by a contest organizer who can flexibly reward contestants based on their numbers of winning battles.[6] For this purpose, we consider dynamic three-battle contests that allow for a whole spectrum of contest technology in the Tullock family, and fully characterize the effort-maximizing prize structures among all feasible allocation rules that are contingent on battle outcomes. Sequential battles can be played either between two fixed players or by varying pairs of players from two opposing teams. We find that the optimal design is a best-of-three contest with both a contest prize to the grand winner of the whole contest and uniform battle prizes to battle winners. The player structure and discriminatory power of the contest technology play crucial roles in determining the prizes' optimal shares.

We first study the optimal contingent prize-allocation rule that elicits the maximum aggregate effort in a sequential-play multi-battle contest between two risk-neutral players with unit marginal effort cost. In every component battle, both players observe the outcomes of previous battles and

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1-prize-money/; http://en.wikipedia.org/wiki/2013_Formula_One_season/; accessed on May 28th, 2015. Similarly, the Professional Golfers' Association(PGA) tour awards winners of all component tournaments, and a grand prize is awarded to the overall best performer at the end of the tour. Refer to http://en.wikipedia.org/wiki/PGA_Tour, accessed on May 28th, 2015.

[5] The contest outcome, however, is eventually determined by battle outcomes. In this sense, there is no loss of generality to describe all feasible prize structures as prize-allocation rules that are solely contingent on battle outcomes.

[6] Gradstein and Konrad (1999) point out that: "contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders."

exert effort simultaneously. We allow a full spectrum of contest technology in the Tullock family to model component battles, which are indexed by the discriminatory power (r) of the corresponding contest success function. Specifically, the contest organizer has a fixed budget (normalized as 1) to fund nonnegative prizes for competing parties.[7] She has the flexibility of fully allocating the budget contingent on battle outcomes, i.e., the wins that each party secures, subject to a monotonicity condition that requires the more battles a party wins, the larger share of prize the party takes.[8] A particularly interesting, but rather intricate, issue is how much prize should be granted to a player with a single win when the discriminatory power (r) of the corresponding Tullock contest technology alters, as we will illustrate in our analysis.

We fully characterize the optimal contingent prize allocation for every positive discriminatory power $r(> 0)$. For each r, we first characterize the subgame perfect equilibrium by backward induction for each eligible contingent prize allocation rule. We then compare across all eligible contingent prize-allocation rules to identify the optimal rule. The procedure requires lengthy computations and multi-step comparison. In particular, computing players' total expected effort for a given prize structure requires aggregating their efforts across every possible path. Because of the difficulties generated by potential ex post asymmetry due to the sequential nature of the contest and its effect on players' strategies, we have to consider multiple overlapping subsets of eligible prize structures separately, and obtain the optimal prize-allocation rule within each subset. Comparison across all the restricted optimums yields the globally optimal prize-allocation rule.

For a sequentially played three-battle individual contest, we find that its optimal prize-allocation rule crucially depends on the discriminatory power r of the contest technology adopted for component battles. The discriminatory power in a Tullock contest measures the importance of a player's effort in determining his winning probability. A higher discriminatory power r means that the winning chances are determined more by players' effort than by other random factors that also affect contestants' performance.[9] Specifically, when the discriminatory power is low, a winner-take-all best-of-three contest is optimal; when the discriminatory power falls in the intermediate or high range, the optimal design takes the form of a best-of-three contest with both a contest prize to the grand winner and uniform battle prizes to battle winners. In particular, in the intermediate range, the battle prize becomes positive and increases to one third of the total prize as the discriminatory power increases. In other words, the optimal prize structure evolves from winner-take-all to proportional division rule as r increases in this range. In the high range, interestingly, a whole span of battle prizes, ranging from winner-take-all to proportional division rule, is optimal.

The economics and intuitions behind these characterizations can be illustrated as follows. For convenience, we use $v(n)$ to denote the prize awarded to a player winning $n \in \{0, 1, 2, 3\}$ battles. It is natural that $v(0) = 0$ (and thus $v(3) = 1$) is necessary to elicit maximal effort from players, since rewarding a player without a single win dampens players' incentive certainly. The more interesting

[7] If the organizer's budget is indivisible, she can equivalently use winning probabilities as design instruments. For convenience, we assume that the prize budget must be exhausted.

[8] We assume that the prize allocation does not depend on the identities of the competing parties.

[9] Fu and Lu (2012a) provide a microfoundation for nested Tullock contests from a noisy ranking perspective.

and intricate trade-off lies in the balance between $v(1)$ and $v(2)$, the prizes for a single win and two wins. Should a positive prize be granted to a player with a single win? If yes, what is the optimal prize level? How should it depend on the discriminatory power r?

We first introduce a useful fact to illustrate the impact of a marginal change in $v(1)$: Any eligible prize structure $\{v(n), n = 0, 1, 2, 3\}$ with $v(0) = 0$ and $v(3) = 1$ is equivalent to the combination of a grand contest prize v_g to the grand winner who wins at least two battles, and a uniform battle prize v_b to the winner of each battle, where $v_b = v(1)$ and $v_g = 1 - 3v(1)$.[10] Therefore, the trade-off between $v(1)$ and $v(2)$ becomes the trade-off between battle prizes and contest prize. A $\Delta(> 0)$ increase in battle prize v_b means a three-time drop in contest prize v_g. Because of the fixed budget, we have to evaluate which prize contributes more in effort elicitation. Intuitively, both grand prize and battle prize can affect players' incentives and effort supply through multiple channels.

We discuss their contributions separately. High battle prizes raise players' effective prize spreads in component battles, and therefore increase players' effort supply in each component battle. In addition, high battle prizes can reduce the momentum/discouragement effect well-established in sequentially played multi-battle contests between two individuals and balance the whole competition. The "strategic momentum effect" or "discouragement effect" as first identified by Harris and Vickers (1987) says that one's (perhaps purely accidental) early lead would allow him to attain easy wins in the future, as it forces his lagging opponent to concede prematurely. One extreme example of battle prizes is the proportional division prize allocation rule, in which each battle winner wins one third of the budget. Given the proportional division rule, since players' prize spreads always equal one third in each battle, early winning will not increase a players' incentive in competing in future battles, i.e., the momentum/discouragement effect disappears, and therefore the competition will not become less competitive after the first or the second battle is played, no matter what the previous outcomes are. On the other hand, a prize structure with high grand prize raises players' effort significantly in the first stage, as well as, the third stage when each player wins one battle. For example, consider a winner-take-all prize structure where the grand contest prize is set at its maximum. Compared to propositional rule, winner-take-all accelerates the end of the competition for two reasons. One is, in a possible situation that each player has won the first two battles, neither player has incentive to fight in battle 3, since no further prize is provided. Moreover, winner-take-all strengthens the momentum/discouragement effect, which means the winner of the first battle has higher incentive and higher chance to win battle 2, due to the desirable grand prize, and therefore the contest will more likely end after the first two battles.[11]

We emphasize that the discriminatory power r plays an important role in accessing these prize structures, in terms of their induced effort levels. Note that under a higher discriminatory power r, effort is more effective in determining the winner, and therefore players react more sensitively to prizes. When discriminatory power r stays low, since other random factors are non-negligible, momentum/discouragement effect is weak. Therefore, with high chance, the contest will not end soon even under winner-take-all structure. In this case, grand prize is more likely to elicit larger effort

[10] This equivalence is established in Section 3.3.

[11] We present more detailed intuitions in Section 3.1.2.

than battle prizes, which leads to the optimality of a maximum grand prize and zero battle prizes. As the discriminatory power r moves into the intermediate range, momentum/discouragement effect becomes stronger. To balance the second-stage contest and elongate the competition, it is optimal to provide a positive battle prize that increases with r. When r moves into a higher range, i.e., $r \geq 2$, players react so sensitively to prizes that their rents are fully dissipated under any eligible prize structure.

For the purpose of comparison and conviction of our explanation above, we further analyze sequential-play three-battle team contests. The multi-battle team contests have been analyzed by Fu, Lu, and Pan (2015). In their model, two teams with an equal number of players compete in a contest. Players from rival teams form pairwise matches to fight in multiple component battles sequentially. A team wins if and only if its players secure a majority number of victories. Each player benefits from his team's win, while he can also receive a private reward for winning his own battle. The authors find that the strategic momentum/discouragement effect typically identified in dynamic multi-battle contests with two players is nullified in this team-contest setting. Häfner (2015) further confirms the robustness of this finding in a tug-of-war team contest with sequential battles. In our model, the prize awarded to a team is a public good for every member of the team, and the organizer endogenously chooses the optimal contingent rule to reward the two teams based on their number of winning battles. We find a winner-take-all best-of-three contest is optimal in this environment for any positive discriminatory power r. The intuition behind the result is rather clear: Even when there is no battle prize, the momentum/discouragement effect does not exist in team contests. In team contests, therefore, a positive battle prize no longer boosts effort supply through the channel of mitigating the momentum/discouragement effect. As a result, a zero battle prize is optimal. The optimality of zero battle prize in team contests further confirms that the optimality of a positive battle prize in contests between the same two individual players is mainly due to mitigation of the discouragement effect.

Our paper primarily belongs to the well-established literature on multi-battle contests. Environments in which the battles are contested sequentially have been analyzed by Harris and Vickers (1987), Ferall and Smith (1999), Klumpp and Polborn (2006), Konrad and Kovenock (2009, 2010), McFall, Knoeber, and Thurman (2009), Malueg and Yates (2010), and Sela (2011), among others. Harris and Vickers (1987) study a multi-battle patent race. Klumpp and Polborn (2006) model U.S. presidential primaries as a multi-battle dynamic contest between two candidates. Malueg and Yates (2010) study players' strategic effort supply in best-of-three contests and test their theoretical prediction empirically using tennis data. All of these studies identify the so-called strategic momentum/discouragement effect in dynamic multi-battle contests with two players. Konrad and Kovenock (2009) completely characterize the unique subgame perfect equilibrium in multi-battle contests with intermediate prizes, in which component contests are modeled as all-pay auctions. They find that even a large lead by one player may not fully discourage the other when a component battle awards a positive intermediate prize.[12] Sela (2011) compares the best-of-three all-pay

[12] Irfanoglua et al. (2011) and Mago and Sheremeta (2012) test these theoretical implications experimentally.

auction to the standard one-stage all-pay auction.

We further this line of research by examining how intermediate prizes can be optimally designed and employed to mitigate the strategic-momentum effect and provide the best incentive for contestants. To our best knowledge, ours is the first study of optimal contingent prize allocation in the context of dynamic multi-battle contests. A winner-take-all prize structure, together with a "majority" winning rule, is commonly adopted in practice in multi-battle contests, and typically assumed in the literature. Our paper rationalizes the optimality of this popular prize-allocation rule in a wide range of environments in a dynamic multi-battle contest context. This finding extends the validity of the winner-take-all principle which has been established in many other settings, including Krishna and Morgan (1998), Moldovanu and Sela (2001) among others. In particular, many papers adopting Tullock contest technology establish the optimality of winner-take-all when discriminatory power r is in low range in various contest settings, such as Clark and Riis (1998), Fu and Lu (2012b), Möller (2012), Schweinzer and Segev (2012), and Clark, Nilssen and Sand (2012) and so on.

We find that a prize-allocation rule that rewards every positive number of wins (i.e., split awards contingent on overall battle outcomes) generates the maximum expected total effort when the battles proceed sequentially between the same two players and the discriminatory power r is either in middle range or high range. In particular, the reward for winning one battle increases and the reward for winning two battles decreases with r when it is in the middle range. These findings suggest that intermediate prizes can facilitate effort elicitation on many occasions in dynamic multi-battle contests between two players.[13]

The merit of a split-award has also been extensively investigated in the procurement literature, where the central issue is the buyer's optimal choice between single sourcing and multiple sourcing. Riordan and Sappington (1989) show that in a dynamic setting, second sourcing can be optimal. Anton and Yao (1992), and Anton, Brusco, and Lopomo (2010) establish the existence of split-award equilibria when a dis-economy of scale is present in production. Gong, Li, and McAfee (2012) rationalize the optimality of a split-award by suppliers' investment incentive. Our study extends this line of research on split-award to the context of dynamic multi-battle contests.

The rest of the paper proceeds as follows. In Section 2, we first set up our model of sequentially played three-battle contests between two same players and introduce prize allocation rules contingent on battle outcomes. We then introduce existing results on equilibrium analysis in single-stage settings, which lay the foundation for analyzing the sequential three-battle contests. In Section 3, we proceed to analyze the sequential three-battle contest under any eligible contest technology and prize allocation rule, and identify the optimal contingent prize-allocation rule for each contest technology within the tullock family. In Section 4, we examine an alternative setting of sequential team contests with multiple pairwise battles, which further illustrates the economic forces that shape the players' incentives and optimal prize allocation rules in different environments. Section 5 provides concluding remarks. The technical proofs are relegated to the appendix and the online

[13] Mago, Sheremeta, and Yates (2013) find that rewarding intermediate battle prizes could boost effort supply in dynamic contests, even with small r. However, in their study, extra battle prizes are funded by an additional budget.

appendix.

2 The model setup

Two players A and B compete in a dynamic three-battle contest. Both of them are risk-neutral and have unit marginal effort cost. They fight the three battles sequentially, and observe the past outcomes (i.e., the state of the contest) before exerting effort in the current battle.

The contest organizer has a prize budget V, which is normalized to 1. The organizer's prize-allocation rule is contingent on the contest outcome, i.e., the number of battles each player wins. Let $v(n)$, $n \in \{0,1,2,3\}$ denote the prize that a player wins if he wins n battles. Alternatively, $v(n)$ can be interpreted as the winning probability of a single indivisible grand prize with value 1. We therefore have the following feasibility restrictions on the prize allocations:

$$v(n) + v(n') = 1, \forall n, n' \in \{0,1,2,3\}, n + n' = 3; \tag{1}$$

$$v(n) \geq v(n'), \forall n \geq n'; \tag{2}$$

$$v(n) \geq 0, \forall n. \tag{3}$$

The first constraint says that the sum of prizes for the two players equals the total prize budget. The second constraint says that the player with more wins is awarded a higher prize. The third constraint is that the prizes cannot be negative, which is natural when $v(n)$ is interpreted as winning probabilities or the players are subject to limited liability. Note that the winning-probability interpretation is particularly relevant when the prize V is indivisible.

In Section 3.3, we will establish that any prize structure $\{v(n), n = 0,1,2,3\}$ is equivalent to a combination of an entry fee/subsidy for each player, a uniform battle prize for the winner of each battle, and a grand contest prize for the player who wins at least two battles. For the ease of exposition, we will first focus on structures $v(n)$, $n \in \{0,1,2,3\}$ in our main analysis, and later adopt the equivalent alternative structures to further interpret the findings.

A generalized Tullock contest technology is adopted for each component battle, in which both players exert their effort simultaneously. Let x_A and x_B denote the players' effort in a battle. Player i's probability of winning the battle is specified by $p_i = \frac{x_i^r}{x_i^r + x_j^r}$ where $i, j \in \{A, B\}$ and $r \in (0, \infty)$ denotes the discriminatory power of the Tullock contest. If $x_A = x_B = 0$, $p_i = \frac{1}{2}$ for any $r \in (0, \infty)$.

Since the battles are played sequentially, the outcome of past battles, or the state of the contest, is observed by both players before the subsequent battle is fought. The state of the contest is denoted by (n_A, n_B) when player $i \in \{A, B\}$ has secured n_i wins. When the state of the contest is (n_A, n_B), we denote player i's effective prize spread of winning the subsequent battle (i.e., the $(n_A+n_B+1)-$th battle) by $v_i(n_A, n_B)$, which is his expected reward of winning the $(n_A+n_B+1)-$th battle; player i's equilibrium effort supply in the subsequent battle by $x_i(n_A, n_B)$; and player i's equilibrium winning probability of the subsequent battle by $p_i(n_A, n_B)$.

In this paper, we study the optimal prize-allocation rule that elicits the highest expected aggregate effort in the contest.

We first present some existing results of equilibrium analysis in a two-player Tullock contest with asymmetric values and an arbitrary discriminatory power r, which will form the foundation for our analysis. Consider two players i and j competing in a generalized Tullock contest with discriminatory power r. The value of player i is v_i and the value of player j is v_j, without loss of generality, assuming $v_i \geq v_j > 0$.[14] Player i's winning probability is given by $p_i = \frac{x_i^r}{x_i^r + x_j^r}$ where x_i and x_j denote players' efforts, and $r \in (0, \infty)$ denotes the discriminatory power of the contest. We let $x_i(v_i, v_j; r)$ and $x_j(v_i, v_j; r)$ denote the players' equilibrium strategy, which can be either pure or mixed.

Definition 1 *For $z \in (0, 1]$, a cutoff $\hat{r}(z) \in (1, 2]$ is defined as the unique solution to $r = 1 + z^r$.*

Nti (1999) establishes that a pure-strategy equilibrium exists if and only if r is bounded from above by a cutoff $\hat{r}(\frac{v_j}{v_i}) \leq 2$ and provides a complete characterization of the equilibrium strategy. Wang (2010) analyzes the case of $r \in (\hat{r}(\frac{v_j}{v_i}), 2]$ and obtains a closed-form solution to the equilibrium strategy.[15] The mixed-strategy equilibrium in an all-pay auction has been analyzed extensively in the literature starting from Hillman and Riley (1989). Alcalde and Dahm (2010) analyze the case of $r > 2$ relying on the result of Baye, Kovenock, and de Vries (1994). They show there exists an "all-pay-auction" equilibrium in mixed strategies, although a closed-form solution of the equilibrium strategies is yet to be identified. We use this result to obtain the equilibrium effort outlays in contests. These characterizations are summarized as follows:

Lemma 1 *Assuming $v_i \geq v_j > 0$, the equilibrium bidding strategies $x_i(v_i, v_j; r)$ and $x_j(v_i, v_j; r)$ are:*
(i) If $r \leq \hat{r}(\frac{v_j}{v_i})$,

$$x_i(v_i, v_j; r) = \frac{r v_i^{r+1} v_j^r}{(v_i^r + v_j^r)^2}, x_j(v_i, v_j; r) = \frac{r v_j^{r+1} v_i^r}{(v_i^r + v_j^r)^2}.$$

(ii) If $r \in (\hat{r}(\frac{v_j}{v_i}), 2]$,

$$x_i(v_i, v_j; r) = (\frac{1}{r-1})^{\frac{1}{r}}(1 - \frac{1}{r})v_j, \ x_j(v_i, v_j; r) = \begin{cases} (1 - \frac{1}{r})v_j, & \textit{with probability } q = \frac{v_j}{v_i}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \textit{with probability } 1 - q. \end{cases}$$

(iii) If $r > 2$,

$$x_i(v_i, v_j; r) = \mu^*, \ x_j(v_i, v_j; r) = \begin{cases} \mu^*, & \textit{with probability } q = \frac{v_j}{v_i}, \\ 0, & \textit{with probability } 1 - q. \end{cases}$$

where μ^* is the (symmetric) equilibrium mixed strategy identified by Baye, Kovenock, and de Vries (1994) in a two-player Tullock contest with $r > 2$, and fully dissipates the rent in the symmetric game when both valuations equal v_j.

[14] Otherwise, we could relabel the two players.

[15] The cutoff $\hat{r}(\frac{y}{x})$ converges to 2 when x approaches y, i.e., when the two players are symmetric. In that case, the particular case analyzed by Wang (2010) vanishes.

In all three cases, $x_i(v_i, v_j; r)$ is a more aggressive strategy than $x_j(v_i, v_j; r)$ as long as $v_i \geq v_j$, i.e., the contestant with the higher valuation tends to exert more effort. In Case (i), Nti (1999) shows that the proposed pure-strategy equilibrium is unique.[16] The uniqueness of the equilibrium in Case (ii) can be shown among the class of semi-pure strategy equilibria in which active players bid in pure strategy, and the uniqueness of equilibrium in Case (iii) is established by Ewerhart (2012). These results together mean that in our setting with sequential battles, the subgame perfect equilibrium for any prize structure and any power r is unique.

To pin down the optimal prize structure that elicits the maximum expected aggregate effort, for each feasible $\{v(0), v(1), v(2), v(3)\}$, we need to compute players' equilibrium efforts along each possible path and the probability of each path. Note that the equilibrium bidding strategies in each battle above depend on $\hat{r}(\frac{v_j}{v_i})$, which varies with the ratio of the two players' prize spreads in the concerned battle. In the next section, we will show that, in an individual contest, players have symmetric prize spreads in the first and third battles, while, in general, their prize spreads are asymmetric in the second battle. Consequently, we have $\hat{r} = 2$ for the first and third battles, and $\hat{r} \in (1, 2)$ in general for the second battle.

3 Optimal prize allocation

To derive the optimal prize-allocation rule for a fixed r associated with the Tullock technology, we have to compare across all feasible prize structures in terms of the induced aggregate effort. For this purpose, we solve the dynamic game backwards by adopting the building-block equilibrium strategies provided by Lemma 1. We therefore consider Case 1, in which $r \in (0, 2]$ in Section 3.1, and Case 2, in which $r > 2$ in Section 3.2. This categorization reflects the fact that different parts of Lemma 1 must be applied in different ranges of r.

We would like to emphasize that in battle 2, since the players' prize spreads are endogenously determined by r and $\{v(0), v(1), v(2), v(3)\}$, and therefore are asymmetric, $\hat{r}$ in Lemma 1 thus must be contingent on prize structure $\{v(0), v(1), v(2), v(3)\}$ and r. As a result, under a prevailing contest technology with the discriminatory power r, different forms of equilibria (as described in Lemma 1) might apply as prize structure varies, which makes it difficult to compare all feasible prize structures to pin down the optimum. An additional technical difficulty is that it is even not straightforward to determine which equilibrium prevails for the given r and $\{v(0), v(1), v(2), v(3)\}$.

Our strategy for searching for the optimal prize structure for any given $r \in (0, \infty)$ is to categorize the prize structures into different overlapping regions. For each region, we are able to pin down the specific expression for the total expected effort and derive the restricted optimum. Then we compare across all regions and obtain the unrestricted global optimum.

[16] Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) show that when $r \in (0, 1]$, there is a unique pure-strategy equilibrium in a one-shot Tullock contest.

3.1 Case 1: $r \in (0, 2]$

We first consider the range of $r \in (0, 2]$, which is more interesting in economics and also more challenging to analyze. As will be shown later, for any feasible prize structure, players' prize spreads are in general asymmetric in the second battle, and therefore the resulting cutoff $\hat{r}$ of Lemma 1 in general falls in $(1, 2)$ for this battle. Depending on the comparison between r and $\hat{r}$, either part (i) or (ii) would apply. Moreover, players' prize spreads depend on the given prize structure and discriminatory power r, which further complicates the comparison between r and $\hat{r}$. To facilitate writing down the aggregate effort clearly as a function of prize structure $\{v(0), v(1), v(2), v(3)\}$ and r, in general we consider four sub-areas for the feasible prize structures for each $r \in (0, 2]$, as will be defined in Definition 3. The details are as follows.

Fix a feasible prize structure $\{v(0), v(1), v(2), v(3)\}$ and an $r \in (0, 2]$. We start from the third stage and look at players' incentives at each reachable state of the whole contest. When each has won one battle, their common effective prize spread of winning the third battle is $v_A(1, 1) = v_B(1, 1) = v(2) - v(1)$. As a result, $\hat{r} = 2$ for battle 3. By Lemma 1(i), their common equilibrium effort supply is given by $x_A(1, 1) = x_B(1, 1) = \frac{r}{4}(v(2) - v(1))$, and each wins the third battle with half probability $p_A(1, 1) = p_B(1, 1) = \frac{1}{2}$. When a player has won the first two battles (i.e., $(n_A, n_B) = (2, 0)$ or $(0, 2)$), he responds to his effective prize spread $v(3) - v(2)$ in battle 3 and his opponent responds to the prize spread $v(1) - v(0)$. The two players have a common effective prize spread of winning the third battle because of the budget constraints (1), which also leads to $\hat{r} = 2$ for battle 3. By Lemma 1(i), their common equilibrium effort supply equals $x_A(2, 0) = x_B(2, 0) = x_A(0, 2) = x_B(0, 2) = \frac{r}{4}(v(1) - v(0))$, and each of them has a half chance to win the third battle. Note that players' incentives remain symmetric in battle 3, regardless of the state.

We now proceed to the second battle. Suppose that the state is $(1, 0)$, i.e., player A has won the first battle. His prize spread equals

$$\begin{aligned} w_A \quad : &= v_A(1,0) = [p_A(2,0)v(3) + p_B(2,0)v(2) - x_A(2,0)] - [p_A(1,1)v(2) + p_B(1,1)v(1) - x_A(1,1)] \\ &= [\frac{1}{2}v(3) + \frac{1}{2}v(2) - \frac{r}{4}(v(1) - v(0))] - [\frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1))] \\ &= (\frac{1}{2} - \frac{r}{4})[1 - v(2) - v(0)] + (\frac{1}{2} + \frac{r}{4})[2v(2) - 1]. \end{aligned}$$

His opponent, player B, the loser of the first battle, has a prize spread

$$\begin{aligned} w_B \quad : &= v_B(1,0) = [p_B(1,1)v(2) + p_A(1,1)v(1) - x_B(1,1)] - [p_B(2,0)v(1) + p_A(2,0)v(0) - x_B(2,0)] \\ &= [\frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1))] - [\frac{1}{2}v(1) - \frac{1}{2}v(0) - \frac{r}{4}(v(1) - v(0))] \\ &= (\frac{1}{2} + \frac{r}{4})[1 - v(2) - v(0)] + (\frac{1}{2} - \frac{r}{4})[2v(2) - 1]. \end{aligned}$$

In the second battle, w_A and w_B, respectively, reflect the incentives of the winner and loser of the first battle when $r \in (0, 2]$. We define them formally below. Note that the two players' incentives in the second battle is in general asymmetric, unless $v(2) - v(1) = v(1) - v(0)$.[17]

DEFINITION 2 *Define* $\eta = \frac{w_B}{w_A}$. *More accurately,* $\eta := \eta(v(0), v(2), r)$ *is function of* $v(0)$, $v(2)$ *and* r.

For all eligible prize profiles, the battle-2 prize-spread ratio $\frac{\min\{w_i,w_j\}}{\max\{w_i,w_j\}}$, which is $\frac{w_A}{w_B}$ or $\frac{w_B}{w_A}$, has a lower bound $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}$ (< 1) for a given $r \in (0, 2]$. In particular, we have $\eta \in [\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, \frac{\frac{1}{2}+\frac{r}{4}}{\frac{1}{2}-\frac{r}{4}}]$ for $r \in (0, 2]$. These bounds will be frequently employed in our analysis.

Given an $r \in (0, 2]$, to solve the players' equilibrium effort supply in battle 2, we adopt the equilibrium strategy described by Lemma 1(i) or 1(ii), and it is the cutoff $\hat{r}(\frac{\min\{w_i,w_j\}}{\max\{w_i,w_j\}})$ that determines which one should be adopted. Note that this cutoff depends on players' incentives to win the second battle, which can be asymmetric and vary with the prize structure $\{v(0), v(1), v(2), v(3)\}$ and r. In fact, it is the prize structure and r that determine the magnitude of each player's incentive to win battle 2, in particular, who has the higher incentive. Consequently, for the given $r \in (0, 2]$, different equilibrium strategies might be applicable in battle 2 as the prize structures vary. To identify the form of the two players' equilibrium strategies in battle 2, we introduce the following four overlapping sets $\mathcal{V}_i, i \in \{0, 1, 2, 3\}$, whose union is the collection $\mathcal{V}$ of all feasible prize-allocation rules. Note that for each set $\mathcal{V}_i$, we are able to pin down the players' equilibrium strategies in the second battle since the associated restrictions make it clear which part of Lemma 1 would apply.

DEFINITION 3 (i) *$\forall r$, we define* $\mathcal{V} = \{(v(0), v(2)) : 0 \le v(0) \le \frac{1}{2} \le v(2) \le 1$ *and* $v(2) + v(0) \le 1\}$, *which is the collection of all feasible prize-allocation rules;*
(ii) $\forall r \in (0, 2]$, $\mathcal{V} = \cup_{i=0}^{3} \mathcal{V}_i$, *where*

$$\begin{aligned}
\mathcal{V}_0 &= \mathcal{V} \cap \{(v(0), v(2)) : \; r \le 1 + (\frac{w_A}{w_B})^r \quad \textit{and} \quad w_A \le w_B\}, \\
\mathcal{V}_1 &= \mathcal{V} \cap \{(v(0), v(2)) : \; r \le 1 + (\frac{w_B}{w_A})^r \quad \textit{and} \quad w_A \ge w_B\}, \\
\mathcal{V}_2 &= \mathcal{V} \cap \{(v(0), v(2)) : \; 1 + (\frac{w_B}{w_A})^r < r \le 2 \quad \textit{and} \quad w_A \ge w_B\}, \\
\mathcal{V}_3 &= \mathcal{V} \cap \{(v(0), v(2)) : \; 1 + (\frac{w_A}{w_B})^r < r \le 2 \quad \textit{and} \quad w_A \le w_B\}.
\end{aligned}$$

The restrictions $0 \le v(0) \le \frac{1}{2} \le v(2) \le 1$ and $v(2) + v(0) \le 1$ in the definition of $\mathcal{V}$ are equivalent to the monotonicity (i.e., (2)) and non-negativity (i.e., (3)) of prizes $v(n)$ under budget constraint (1).

In general, we need to search for the optimal prize structure within each $\mathcal{V}_i$ and compare across all four $\mathcal{V}_i$s. However, we first restrict our search to the prize structures in $\mathcal{V}_0 \cup \mathcal{V}_1$. And then we will show that it is without loss of any generality, since the optimal prize structure is within $\mathcal{V}_0 \cup \mathcal{V}_1$.

[17] The alternative interpretation of the prize structure in Section 3.3 shows that this condition means that each battle provides one third of the total budget to its winner. The rewards, therefore are independent of players' aggregate performance under this condition.

3.1.1 Analysis

We are now ready to analyze the optimal prize structure for Case 1, i.e. when $r \in (0,2)$. For each such prize structure in $\mathcal{V}_0 \cup \mathcal{V}_1$, we can pin down the expected aggregate effort elicited at the subgame perfect equilibrium. The details are presented in the proof of Lemma 2 (see Appendix A). For any such prize structure, computing aggregate effort requires pinning down the players' equilibrium effort at each stage for every possible path. Lemma 1(i) is applicable in all three battles for prize structures in $\mathcal{V}_0 \cup \mathcal{V}_1$. The equilibrium is unique for the reasons illustrated after Lemma 1.

LEMMA 2 *Within $\mathcal{V}_0 \cup \mathcal{V}_1$, the aggregate expected equilibrium effort equals*

$$TE_1 = \frac{r w_A^r w_B^r}{[w_A^r + w_B^r]^2}[(1-\frac{r}{2})w_A + (1+\frac{r}{2})w_B] + \frac{r}{2}[2v(2)-1] + \frac{w_A^r r[1-v(2)-v(0)]}{w_A^r + w_B^r}. \tag{4}$$

Note that TE_1 solely depends on prizes $v(0)$ and $v(2)$, which satisfy the following restrictions: $0 \le v(0) \le \frac{1}{2} \le v(2) \le 1$; and $v(0) + v(2) \le 1$.

We introduce the following definition, which facilitates identifying the net effect of $v(2)$ on the aggregate effort, i.e. $\frac{d}{dv(2)}[TE_1(v(0), v(2))|_{v(0)=0}]$.

DEFINITION 4 *Define*

$$\begin{aligned} D(\eta, r) \quad \equiv \quad & [(\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta][(1-\frac{r}{2})(-r-1+(r-1)\eta^r)+(1+\frac{r}{2})(-r+1+(r+1)\eta^r)\eta] \\ & +2\eta(1+\eta^r)(2-\frac{3r^2}{4}+\eta^r). \end{aligned} \tag{5}$$

As we will show later, $D(\eta, r)$ summarizes the net effect of a change in $v(2)$ on the aggregate effort $TE_1(v(0), v(2))|_{v(0)=0}$. In addition, we define $\underline{r}$ and r^* such that $D(\eta, r)|_{\eta = \frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, r=\underline{r}} = 0$ and $D(\eta, r)|_{\eta=(r-1)^{\frac{1}{r}}, r=r^*} = 0$ with $1 < \underline{r} < r^* < 2$. Cutoff $\overline{r}$ is further defined such that $\frac{\frac{1}{2}-\frac{\overline{r}}{4}}{\frac{1}{2}+\frac{\overline{r}}{4}} = (\overline{r}-1)^{\frac{1}{\overline{r}}}$. And $\overline{r} \in (\underline{r}, r^*)$.[18] In particular, simulations show that $\underline{r} \approx 1.09$, $\overline{r} \approx 1.19$ and $r^* \approx 1.31$.

We now state our main result on optimal prize structure for Case 1 in the following theorem.

THEOREM 1 *In a sequential three-battle contest with two players, the unique effort-maximizing prize structure $(v^*(0), v^*(1), v^*(2), v^*(3))$ is respectively:*

(i) $(0,0,1,1)$ *(i.e., winner-take-all) if $r \in (0, \underline{r}]$;*

(ii) $(0, 1-v_r^*(2), v_r^*(2), 1)$ *if $r \in (\underline{r}, r^*]$, where $v_r^*(2) \in (\frac{2}{3}, 1)$ is the unique $v(2)$ that solves*

$$D(\eta(v(0), v(2), r), r)|_{v(0)=0} = 0;$$

(iii) $(0, 1-v_r^*(2), v_r^*(2), 1)$ *if $r \in (r^*, 2]$, where $v_r^*(2) = 1 - \frac{(\frac{r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{r}{4})(r-1)^{\frac{1}{r}}}{(\frac{3r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{3r}{4})(r-1)^{\frac{1}{r}}} \in [\frac{2}{3}, 1)$ decreases with r and reaches $\frac{2}{3}$ (i.e., proportional rule) when $r = 2$.*

[18] We analytically show that $\underline{r} < \overline{r} < r^*$. Please refer to proofs of Properties 1 and 2 in the online appendix for the argument.

Proof. The proof of Theorem 1 is quite lengthy and consists of several major steps. Here, we only depict a road map. The complete proof is provided in Appendix A.

In step 1, we consider two subcases with $r \in (0, \overline{r}]$ and $r \in (\overline{r}, 2]$ respectively, and show that there is no loss of generality to search for the optimal prize structure within region $\mathcal{V}_0 \cup \mathcal{V}_1$ when $r \leq 2$. We thus can adopt the TE_1 in (4) of Lemma 2 as the applicable total effort for our analysis.

In step 2, we show that $v^*(0) = 0$ and thus $v^*(3) = 1$ at optimum. With the optimal $v^*(0)$ and $v^*(3)$, we further show that the first-order derivative of $TE_1(v(0), v(2))|_{v(0)=0}$ with respect to $v(2)$ shares the same sign with $D(\eta(v(0) = 0, v(2), r), r)$, where $\eta(v(0) = 0, v(2), r) := \frac{w_B}{w_A}|_{v(0)=0} = \frac{(\frac{1}{2}+\frac{r}{4})(1-v(2))+(\frac{1}{2}-\frac{r}{4})(2v(2)-1)}{(\frac{1}{2}-\frac{r}{4})(1-v(2))+(\frac{1}{2}+\frac{r}{4})(2v(2)-1)}$ only depends on $v(2)$ and r. In particular, it decreases with $v(2)$.

In step 3, we introduce Properties 1 to 3 of the $D(\eta, r)$ function in Appendix A, which play instrumental roles in determining $v_r^*(2)$ in the following proofs for parts (i)-(iii) of this theorem. In particular, Properties 1 and 2 imply $\overline{r} \in (\underline{r}, r^*)$, and that $D(\eta, r)|_{\eta=\underline{\eta}_r}$ is U-shaped with two roots at $\underline{r}$ and r^* by construction, and reaches its minimum at $\overline{r}$ (see Figure 1). Here, $\underline{\eta}_r$ is the lower bound that $\eta(v(0) = 0, v(2), r)$ can reach within region $\mathcal{V}_0 \cup \mathcal{V}_1$ for given r. Specifically,
$$\underline{\eta}_r = \begin{cases} \frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, & r \in (0, \overline{r}], \\ (r-1)^{\frac{1}{r}}, & r \in (\overline{r}, 2]. \end{cases}$$[19]

For part (i) where $r \leq \underline{r}$, recall $\eta(v(0) = 0, v(2), r)$ strictly decreases with $v(2) \in [\frac{1}{2}, 1]$, and note $\eta(v(0) = 0, v(2), r)$ reaches its minimum $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}$ when $v(2) = 1$.

When $r \in (0, \underline{r})$, for any $v(2) \in [\frac{1}{2}, 1]$, we have $D(\eta(v(0) = 0, v(2), r), r) > 0$ and thus $v^*(2) = 1$, since $D(\eta(v(0) = 0, v(2), r), r) \geq D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}}$ by Property 3 and $D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}} > D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, r=\underline{r}} = 0$ by Property 1.

When $r = \underline{r}$, $D(\eta(v(0) = 0, v(2), r), r)|_{r=\underline{r}} > D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, r=\underline{r}} = 0$ also holds for any $v(2) \in [\frac{1}{2}, 1)$. We also have $v^*(2) = 1$.

For the proof of part (ii), we consider two ranges: $r \in (\underline{r}, \overline{r}]$ and $r \in (\overline{r}, r^*]$, since two different lower bounds for η would apply.

When $r \in (\underline{r}, \overline{r}]$, we have that the lower bound $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}$ for η still applies. We have $D(\eta(v(0) = 0, v(2), r), r)|_{v(2)=1} = D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}} < 0, \forall r \in (\underline{r}, \overline{r}]$ by Property 1, while $D(\eta(v(0) = 0, v(2), r), r)|_{v(2)=\frac{2}{3}} = 3r^2 + 12 > 0$. Moreover, $D(\eta(v(0) = 0, v(2), r), r)$ decreases in $v(2)$ by Property 3 and the fact that $\eta(v(0) = 0, v(2), r)$ decreases with $v(2)$. Together with the continuity of $D(\eta(v(0) = 0, v(2), r), r)$ in $v(2)$, there exists, for each $r \in (\underline{r}, \overline{r}]$, a unique solution $v_r^*(2) \in [\frac{2}{3}, 1)$ of $D(\eta(v(0) = 0, v(2), r), r) = 0$ such that $D(\eta(v(0) = 0, v(2), r), r) > 0$ when $v(2) < v_r^*(2)$ and $D(\eta(v(0) = 0, v(2), r), r) < 0$ when

[19] Note that the bound $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}$ is achieved by the feasible prize structure $(v(0), v(1), v(2), v(3)) = (0, 0, 1, 1) \in \mathcal{V}_1$ for $r \in (0, \overline{r}]$, and the other bound $(r-1)^{\frac{1}{r}}$ is also achievable within $\mathcal{V}_1$ for $r \in (\overline{r}, 2]$. $\forall r \in (\overline{r}, 2]$, we have $\frac{w_B}{w_A}|_{v(0)=0, v(2)=1} = \frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}} < (r-1)^{\frac{1}{r}}$. Take $v(2; r) \in (\frac{1}{2}, 1)$ such that $\frac{w_B}{w_A}|_{v(0)=0} = 1$. Clearly, $\frac{w_B}{w_A}|_{v(0)=0} \geq (r-1)^{\frac{1}{r}}$. Note that $\frac{w_B}{w_A}|_{v(0)=0}$ drops with $v(2)$. Since $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}} < (r-1)^{\frac{1}{r}}$, we must have $v(2) \in [v(2; r), 1)$ such that $\frac{w_B}{w_A}|_{v(0)=0} = (r-1)^{\frac{1}{r}}$. Note that such a prize profile is in $\mathcal{V}_1$ for $r \in (\overline{r}, 2]$.

$v(2) > v_r^*(2)$. Therefore, the prize structure $(v(0), v(1), v(2), v(3)) = (0, 1 - v_r^*(2), v_r^*(2), 1)$ induces the maximum aggregate effort when $r \in (\underline{r}, \overline{r}]$.

When $r \in (\overline{r}, r^*]$, the associated restrictions $r \leq 1 + (\frac{w_B}{w_A})^r = 1 + \eta^r$ on prize structures $\mathcal{V}_0$ and $\mathcal{V}_1$ implies the lower bound $(r-1)^{\frac{1}{r}}$ applies. The rest of the proof is analogous to the above arguement.

For the proof of part (iii) where $r \in (r^*, 2]$, we have lower bound $(r-1)^{\frac{1}{r}}$ applies and $D(\eta, r) \geq D(\eta, r)|_{\eta=(r-1)^{\frac{1}{r}}} > 0$, where r^* satisfies $D(\eta, r)|_{\eta=(r-1)^{\frac{1}{r}}, r=r^*} = 0$. Therefore, an increase in $v(2)$ always raises the aggregate effort for each $r \in (r^*, 2]$. As a result, the optimal $v_r^*(2)$ is $1 - \frac{(\frac{r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{r}{4})(r-1)^{\frac{1}{r}}}{(\frac{3r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{3r}{4})(r-1)^{\frac{1}{r}}}$, which corresponds to the applicable lower bound $(r-1)^{\frac{1}{r}}$ for η. We have $\frac{d}{dr}(v_r^*(2)) < 0$ for $r \in (r^*, 2]$.[20] ■

Theorem 1 fully identifies the optimal prize structure for every $r \in (0, 2]$: When $r \in (0, \underline{r}]$, winner-take-all is optimal; when $r \in (\underline{r}, r^*]$, the optimal $v_r^*(2)$ is implicitly determined and falls in $(\frac{2}{3}, 1)$, and simulations show that $v_r^*(2)$ strictly decreases in this range;[21] when $r \in (r^*, 2]$, $v_r^*(2) = 1 - \frac{(\frac{r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{r}{4})(r-1)^{\frac{1}{r}}}{(\frac{3r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{3r}{4})(r-1)^{\frac{1}{r}}}$, which decreases with r.

In order to show Theorem 1, we evaluate the net effect of $v(2)$ on the aggregate effort for any $r \in (0, 2]$ by checking $D(\eta, r)$, which summarizes the effect of a change in $v(2)$ on the aggregate effort, and note that $D(\eta, r)$ decreases with $v(2)$, $\forall r \in (0, 2]$.

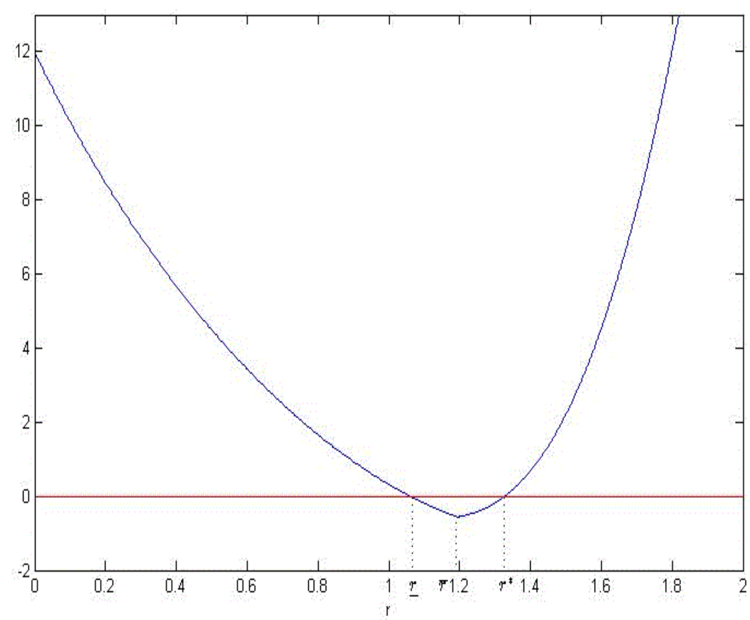

Figure 1: $D(\eta, r)|_{\eta=\underline{\eta}_r}$

[20] This is because $\frac{d}{dr}(v_r^*(2)) = -\frac{4}{r\left(3r+3r(r-1)^{\frac{1}{r}}+2(r-1)^{\frac{1}{r}}-2\right)^2}[r(1-(r-1)^{\frac{2}{r}}) + (r-1)^{\frac{1}{r}}(\frac{2r}{r-1} - 2(\ln(r-1)))] < 0$, $\forall r \in (1, 2)$.

[21] Since $v_r^*(2)$ is implicitly determined in this range, an explicit expression is not available as well as an analytical proof for the monotonicity.

When $D(\eta, r)|_{\eta=\underline{\eta}_r} > 0$ for a particular r, we have $\frac{d}{dv(2)}TE_1(v(0), v(2))|_{v(0)=0} > 0$ for all eligible $v(2)$. That is, an increase in $v(2)$ raises aggregate effort so that $v(2)$ should be set as high as possible at the optimum. In other words, the optimal $v^*(2)$ should be set at the highest applicable level. This is the case when $r \leq \underline{r}$, and therefore the optimal $v_r^*(2)$ is set at the highest value of 1 in applicable sets $\mathcal{V}_0 \cup \mathcal{V}_1$. This is also the case when $r \geq r^* \in (\overline{r}, 2)$. Therefore, for $r \geq r^*$, the optimal $v_r^*(2)$ is set at the highest possible value of $1 - \frac{(\frac{r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{r}{4})(r-1)^{\frac{1}{r}}}{(\frac{3r}{4}-\frac{1}{2})+(\frac{1}{2}+\frac{3r}{4})(r-1)^{\frac{1}{r}}}$ constrainted by prize sets $\mathcal{V}_0 \cup \mathcal{V}_1$ for such an r.

When $D(\eta, r)|_{\eta=\underline{\eta}_r} < 0$ for a particular r, we have $\frac{d}{dv(2)}TE_1(v(0), v(2))|_{v(0)=0} < 0$ for the highest $v(2)$ in the applicable prize sets, so that reducing this $v(2)$ would increase aggregate effort until $D(\eta(v(0), v(2)), r) = 0$. In this case, we have an interior optimum. This is the case when $r \in (\underline{r}, r^*)$. In this case, $v_r^*(2)$ is uniquely determined by $D(\eta(v(0) = 0, v_r^*(2), r), r) = 0$.

We plot $v_r^*(1) = 1 - v_r^*(2)$ as a function of r in Figure 2. The optimal prize structure crucially depends on the discriminatory power r of the contest technology. Specifically, optimal prize $v_r^*(1)$ for a single win remains zero when r is low, i.e. $r \in (0, \underline{r}]$; and then becomes positive when r falls in the intermediate, i.e. $r \in (\underline{r}, r^*]$; when $r \in (r^*, 2]$, $v_r^*(1)$ strictly and continuously increases with r and reaches $\frac{1}{3}$ at the maximum. In sum, the optimal prize structure evolves from winner-take-all to proportional division rule as r increases in the range of $(0, 2]$.

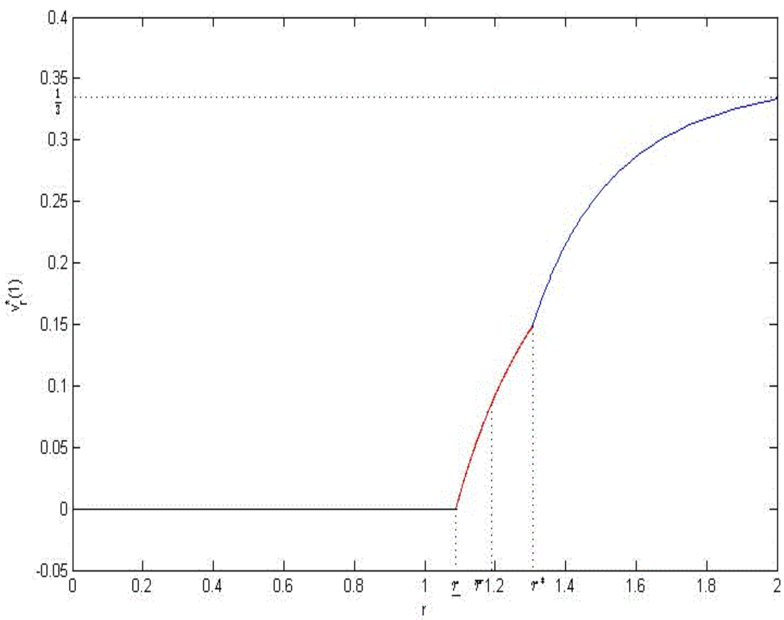

Figure 2: The Optimal Prize $v_r^*(1)$

3.1.2 Intuitions

First of all, it is natural that the optimal $v^*(0) = 0$ and thus $v^*(3) = 1$ for every $r \in (0, 2]$, since awarding a player who wins no battle and thus panelizing the other player who wins all battles would not provide more incentive for players to exert high effort. We thus focus on understanding how the optimal $v^*(1)$ or equivalently $v^*(2)$ reacts to a change in contest technology (i.e. r). Note that $v(2) \geq \frac{1}{2}$ for any feasible prize structure. Consider a decrease in $v(2)$, and therefore an increase in $v(1)$ (due to the budget constraint (1)), which increases the incentive of the loser of battle 1, but reduces the incentive of the winner of battle 1. Consequently, the contest becomes more balanced in the sense that the early loser is more likely to win the second battle. As a result, both players are better incentivized in battle 2. However, both players tend to exert less effort in battle 1 to fight for the reduced advantage in the second battle.

The mitigated momentum/discouragement effect in the second battle due to the marginal increase in $v(1)$ increases the chance that the contest will reach a state $(1, 1)$ and decreases the chance of state $(2, 0)$ or $(0, 2)$ for the third battle, which tends to increase the effort supply in the third battle. This is because at state $(1, 1)$, the competition is more balanced, and therefore elicits higher effort from players. However, a negative effect in battle 3 can also arise from a higher $v(1)$. An increase in $v(1)$ reduces the prize spread in the third battle when state $(1, 1)$ is reached, and enlarges the prize spread in the third battle when state $(2, 0)$ or $(0, 2)$ is reached. Specifically, the decrease in prize spread for state $(1, 1)$ doubles the increase in prize spreads for states $(2, 0)$ and $(0, 2)$.

Nevertheless, when the prevailing momentum/discouragement effect in the second battle is sufficiently strong due to the higher r, an increase in $v(1)$ does not only encourage the early loser and thus also better incentivize the early winner when the contest reaches battle 2, but also boost the effort supply in the third battle through significantly increasing the probabilities that state $(1, 1)$ will be reached. As a result, for effort elicitation purpose, the higher the r is, the higher prize $v(1)$ should be in order to offset the stronger momentum/discouragement effect and better balance the contest.

The magnitudes of the above positive and negative effects all depend on contest technology r. In particular, an increase in $v(1)$ is more effective in mitigating the momentum effect in the second battle when the contest technology gets more discriminatory (i.e., r increases). We therefore check $D(\eta, r)$ to evaluate carefully the net effect of $v(2)$ on the aggregate effort for any $r \in (0, 2]$ for a clear trade-off.

3.2 Case 2: $r > 2$

To consider the remaining case of $r > 2$, analogous to Definition 3(*ii*) for Case 1, we introduce in the following Definition 5 prize subsets $\mathcal{V}_4$ and $\mathcal{V}_5$, whose union equals the whole set $\mathcal{V}$. Considering $\mathcal{V}_4$ and $\mathcal{V}_5$ separately allows us determine which player has higher incentive to win battle 2 so that the corresponding equilibrium given by Lemma 1(*iii*) can be adopted for solving battle 2.

Definition 5 *$\forall r > 2$, $\mathcal{V} = \cup_{i=4}^{5} \mathcal{V}_i$, where $\mathcal{V}_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\}$ and $\mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\}$.*

In Lemmas 6 and 7 (in Appendix B), we provide the expected aggregate effort and then characterize the optimal prizes in $\mathcal{V}_4$ and $\mathcal{V}_5$, respectively. Combining the results, we can identify the optimal prize structures for each $r > 2$. Therefore, we have Theorem 2 in the following.

THEOREM 2 *When $r > 2$, the maximum aggregate effort of* 1 *is induced by any prize structure* $(0, 1 - v(2), v(2), 1)$, $\forall v(2) \in [\frac{2}{3}, 1]$.

Theorem 2 illustrates that for the high range of $r \in (2, \infty)$, a whole span of battle prizes, ranging from winner-take-all to proportional division rule, is optimal. This result is in fact intuitive. When r moves into a higher range, i.e., $r \geq 2$, players react so sensitively to incentive that their rents are fully dissipated under any eligible prize structure.

3.3 An alternative interpretation: battle prize and grand contest prize

The optimal prize structures described above can be conveniently interpreted using uniform battle prizes and a grand contest prize. A player gets a battle prize v_b whenever he wins an individual battle, and a grand winner who wins at least two battles gets a grand contest prize or a punishment v_g.[22] We also allow an entry subsidy v_0 to both players. Under this alternative prize structure, a player receives a prize v_0 if he wins no battle; he receives a prize $v_0 + v_b$ is he wins a single battle; he receives a prize $v_0 + 2v_b + v_g$ if he wins two battles; and he receives a prize $v_0 + 3v_b + v_g$ if he wins all three battles.

For any eligible prize structure $\{v(0); v(1); v(2); v(3)\}$ that satisfies restrictions $(1), (2)$ and (3) in the original setup, we next identify a unique prize structure (v_0, v_b, v_g) that satisfies budget constraint $3v_b + v_g + 2v_0 = 1$ and generates the identical effective rewards for every possible contingency of contest states. Let

$$v(0) = v_0, \tag{6}$$

$$v(1) = v_0 + v_b, \tag{7}$$

$$v(2) = v_0 + 2v_b + v_g, \text{ and} \tag{8}$$

$$v(3) = v_0 + 3v_b + v_g. \tag{9}$$

Solving this linear equation system, we obtain that $v_0 = v(0)$, $v_b = v(1) - v(0)$ and $v_g = v(2) - v(1) - (v(1) - v(0))$, which satisfies automatically the budget constraint: $2v_0 + 3v_b + v_g = 2v(0) + 3(v(1) - v(0)) + v(2) - v(1) - (v(1) - v(0)) = v(2) + v(1) = 1$.[23]

Since each player faces the same prize spread across the two prize structures at every state of the contest, each player exerts the same effort across the two setups at each contingency along the path of the contest. We thus establish the equivalence between any eligible prize structure $\{v(n), n = 0, 1, 2, 3\}$ and the combination of a entry fee/subsidy v_0, a grand contest prize v_g, and uniform battle prizes v_b with $v_0 = v(0)$, $v_b = v(1) - v(0)$, and $v_g = 1 - 3v(1) + v(0)$. And note that this equivalence result has nothing to do with the optimality of the prize structure.

[22] We allow a negative contest prize v_g.

[23] Note that when $v(0) = 0$, we have $v_g < 0$ if and only if $v(1) > \frac{1}{3}$, but it never occurs for the optimal prize allocation when $v(1) \leq \frac{1}{3}$.

Applying the equivalence result above, we therefore have the following implementation result for the optimal prize structure identified in Theorems 1 and 2.

THEOREM 3 $\forall r > 0$, *the optimal design can be implemented by a sequential best-of-three contest with a uniform battle prize* $v_b^*(r) = v_r^*(1) \leq \frac{1}{3}$ *to each battle winner and a grand contest prize* $v_g^*(r) = 1 - 3v_r^*(1) \geq 0$ *to the grand winner who wins at least two battles.*

In this alternative setting, we could rewrite the economic intuitions behind Theorems 1 and 2 as follows. Note that optimality of $v^*(0) = 0$ corresponds to $v_0 = 0$, the main trade-off therefore lies between the battle prizes and the grand prize. More precisely, a $\Delta(> 0)$ increase in battle prize v_b comes together with a three-time drop in contest prize v_g. To better understand the optimal allocations, we illustrate how battle prizes and grand prize contribute to aggregate effort respectively. To begin with, suppose there is an increase in battle prizes, it could raise effort supply through multiple channels: First, it directly contributes to higher effective prize spreads in component battles, which therefore induce higher effort supply in each battle. Moreover, such change will cause "ripple effects" on players' effort supply in later stages: In battle 2 where players' incentives are asymmetric, a large battle prize encourages the low-incentive player, the loser of the first battle, to fight. On one hand, this in turn raises the effort from the winner of the first battle in battle 2 as well. On the other hand, this mitigates the well-established discourage effect so that the contest will be more balance in battle 2, and then the grand winner will be more likely determined only after the third battle has been fought. Consequently, the third-stage contest becomes more likely competitive, due to the grand prize.

We now turn to the effects of the grand prize. Fix the path of the contest, an increase in the grand contest prize raises players' effort supply in all stages, especially for the stage contest in which both players have symmetric incentives, e.g., the first stage where the state is $(0,0)$ and third stage when the state is $(1,1)$. At $(0,0)$, a large grand prize increases players' effort indirectly through increasing players' incentive to fight for the status, for being the winner of the first battle who will more likely win the grand prize. A grand prize contributes directly at the state $(1,1)$ through increasing players' prize spreads three-time more than battle prizes. However, changing grand prize affects the path of the whole contest. A high grand prize strengthens the momentum/discouragement effect, since the winner of the first battle will be motivated by a large grand prize. Consequently, the contest will be more imbalanced in second stage, and therefore the contest then has less chance to reach $(1,1)$, at which grand prize is significantly effective. In sum, a high grand prize makes itself less contributive in late stages because it favours imbalanced states, $(2,0)$ or $(0,2)$, meanwhile, battle prizes can mitigate the momentum/discouragement effect to balance the contest.

We emphasize that the higher discriminatory power r is, the more mitigation is needed in battle 2 to offset the stronger momentum/discouragement effect, and the stronger the mitigation provided by battle prize is. Higher battle prize thus serves the purpose. Note that under a higher discriminatory power r, effort is more effective in determining the winner, and therefore players react more sensitively to prizes.

When discriminatory power r stays low, momentum or discouragement effect is weaker. There is no need to provide battle prizes to mitigate discouragement effect for effort elicitation. Even without the battle prizes, the contest will more likely reach state $(1, 1)$, where the grand prize plays its role. Even with battle prizes, especially in the second-stage contest, players react insensitively, and therefore the mitigation of discouragement effect is still weak. In this case, grand prize dominates, which results in the optimality of a maximum grand prize with zero battle prizes. As the discriminatory power r moves into the intermediate range, battle prizes get more important and effective in mitigating discourage effect in battle 2, which renders the optimality of positive battle prizes that increase with r. When r falls into the higher range, i.e., $r \geq 2$, we have a special situation, where rents are fully dissipated in the first battle, in which the two players' prize spreads are symmetric. As a result, any eligible prize structure yields the same level of total effort supply.

Based on Theorems 1 to 3, we have the following observation regarding the optimal prize-allocation rule, which says that the proportional-division rule is, in general, not optimal. In other words, in multi-battle contests, the optimal prize-allocation rule should in principle give an additional reward to the grand winner of the whole contest. In addition, the magnitude of this additional reward (or grand prize) varies with the discriminatory power r of the corresponding contest technology: When $r < 2$, i.e., players' effort is not effective, the lower r is, the higher the addition reward $v_g^*(r)$ should be; when $r \geq 2$, players' effort is so effective that organizer can fully extract their surplus by either battle prizes or grand prize, in this case, any feasible prize structure $(v_g^*(r), v_b^*(r))$ with $v_g^*(r) \in [0, 1]$ is optimal, and feasibility implies that $v_b^*(r) = \frac{1-v_g^*(r)}{3}$.

Corollary 1 $\forall r > 0$, *we always have the optimal grand prize* $v_g^*(r) \in [0, 1]$ *and battle prizes* $v_b^*(r) \in [0, \frac{1}{3}]$. *Moreover,* $v_g^*(r)$ *evolves gradually from* 1 *to* 0, *and* $v_b^*(r)$ *evolves gradually from* 0 *to* $\frac{1}{3}$ *when* $r \in (0, 2)$; *any* $(v_g^*(r), v_b^*(r))$ *with* $v_g^*(r) \in [0, 1]$ *and* $v_b^*(r) = \frac{1-v_g^*(r)}{3}$ *is optimal when* $r \geq 2$.

4 Team contests with pairwise battles

In Section 3, we analyzed a three-battle contest between two players. We illustrated that the need to mitigate the momentum/discouragement effect in such environment plays a crucial role in determining the optimal prize structure. Battle prizes are optimal only if the discriminatory power r of the contest technology becomes relatively high such that the momentum/discouragement effect looms large. In this section, we reinforce this point by further studying an alternative environment with sequential battles, where the momentum/discouragement effect, however, does not exist.

In this section, we study a three-battle contest with two teams of three players each. Battles are sequentially played, and history is observed by all players. Each battle is fought between two players from opposing teams, and each player from a team plays only one battle. Different from dynamic contests between two same players, in this team contest, each team member cares only about the final reward and his own effort cost in his own battle. The prize from the team's winning the contest is assumed to be a public good among team members.

This environment of team contests with multiple pairwise battles was first studied by Fu, Lu, and Pan (2015) for the purpose of equilibrium analysis of a large class of contest technologies of homogeneity-of-degree-zero while adopting the winner-take-all prize-allocation rule. Häfner (2015) further carries out the equilibrium analysis in a tug-of-war variant of this environment while assuming all-pay-auction technology and a winner-take-all prize-allocation rule.

We adopt the same prize-allocation framework as specified in Section 2: $v(n)$, $n \in \{0, 1, 2, 3\}$ denotes the prize allocated to each team based on their number of wins. All three players in a team evaluate the prize at $v(n)$, as the prize is a public good within a team. Alternatively, the players within a same team can equally split the prize their team wins. This alternative prize-sharing rule does not affect the optimal prize allocation.

Based on the insights of Fu, Lu, and Pan (2015), both involved players have a common prize spread in each battle, regardless of the previous outcome. However, depending on the magnitude of discriminatory power r, the equilibrium bidding strategy takes two different forms. Below we show that a pure equilibrium is described by Lemma 1(i) in the first case for $r \leq 2$, and a mixed equilibrium by Lemma 1(iii) in the second case for $r > 2$. The procedure for deriving the unique subgame perfect equilibrium is standard.[24] We show that, when $r \leq 2$, the expected total effort is $TE_T^1 = \frac{3r}{4}(v(2) - v(0))$ in the team contest in Lemma 8 (in Appendix C), and when $r > 2$, the expected total effort is $TE_T^2 = \frac{3}{2}(v(2) - v(0))$ in the team contest in Lemma 9 (in Appendix C). Maximizing the total effort given by Lemmas 8 and 9 under the required constraints for $v(n)$ yields the optimal prize structure $(v(0), v(1), v(2), v(3)) = (0, 0, 1, 1)$. Therefore, we have the following result.

Theorem 4 *Winner-take-all is the unique optimal prize structure in dynamic team contest with pairwise battles.*

We thus conclude that the optimal prize-allocation rule can be dramatically different across contests played between the same two players and two teams of multiple players. When the discriminatory power r is low (i.e., $r \leq \underline{r}$), both types of contests require winner-take-all as the optimal prize-allocation rule. When discriminatory power r is in the middle range (i.e., $r \in (\underline{r}, 2)$), a team contest still requires a winner-take-all prize allocation for effort maximization, while the contest between two same individual players requires awarding positive prizes to a player who wins even a single battle. When the discriminatory power r is in the high range (i.e. $r \geq 2$), a winner-take-all prize-allocation rule still uniquely maximizes the total effort supply in team contest, while a whole span of prize-allocation rules, ranging from winner-take-all to proportional-division rule, is optimal in a contest between two individual players.

Unlike individual contests studied in Section 3, since momentum/discouragement effect does not appear in the team contests, there is no need to mitigate such effect by awarding positive battle prizes. Our analysis of team contests further reinforces the insight that it is the mitigation of the momentum/discouragement effect that leads to the optimality of positive battle prizes in contests between two individuals.

[24] The uniqueness of the subgame perfect equilibrium is discussed after Lemma 1.

5 Concluding remarks

In this paper, we fully characterize the optimal contingent prize-allocation rule in sequential-play three-battle contests between two same players or two teams with three players each. The full spectrum of contest technologies in the Tullock family are accommodated in our analysis. The optimal design can be implemented by a best-of-three contest with uniform battle prizes and a grand contest prize.

A winner-take-all best-of-three (i.e., a party wins all prize money if he wins more than two battles) induces the maximal total expected effort for contests between teams. When the battles are between two same individual players, the discriminatory power of the contest technology plays a crucial role in determining the optimal prize-allocation rule. Specifically, when discriminatory power is in the low range, a winner-take-all best-of-three contest remains optimal. When the discriminatory power is in the intermediate range, the optimal battle prize becomes positive, and then strictly increases with the discriminatory power and reaches one third at the maximum. In other words, the optimal prize structure evolves gradually and continuously from winner-take-all to proportional division rule as the contest technology becomes more and more discriminatory. When the discriminatory power falls in the high range, a whole span of allocation rules in between winner-take-all and proportional division induces the maximal total expected effort.

The difference in optimal prize structures across the two contest environments reflects the different dynamics in multi-battle contests between two individuals and two teams. A positive prize for a single win (or, equivalently, a battle prize) can be optimal in dynamic multi-battle contests between two individuals mainly because it functions to mitigate the momentum/discouragement effect in such contests. A zero battle prize turns out to be optimal in dynamic contests between two teams, since the momentum/discouragement effect does not exist in such team setting as shown in Fu, Lu and Pan (2015), and therefore there is no need to provide positive battle prize for mitigation.

In this paper, we focus on sequential-play three-battle contests. The insights obtained extend to the design of simultanous-play multi-battle contests. One can verify that a winner-take-all best-of-three remains optimal for simultanous-play three-battle team contests, in which the momentum/discouragement effect is not a concern.[25] For the same reason, one can reasonably expect that a winner-take-all best-of-three contest is optimal when the battles are played simultaneously between two individuals.[26]

Our findings provide a rationale from the perspective of effort elicitation for the commonly adopted winner-take-all prize allocation in dynamic multi-battle contests, as well as the practice of setting intermediate prizes on many occasions in a single integrated model. In particular, our analysis sets an upper bound on the maximum prize for the player with a single win: Its optimal

[25] The symmetric equilibrium in which players adopt the same strategy, is easy to characterize for all discriminatory power $r > 0$.

[26] The technical difficulties of analyzing simultanous-play three-battle contests lie in the equilibrium characterizations of contests between two same players. The procedure of Szentes and Rosenthal (2003) can be adapted for equilibrium construction in all-pay auctions. The method of Klumpp and Polborn (2006) can be extended to analyze equilibrium when component battles are modelled as a Tullock contest with discriminatory power $r \in (0, 1]$.

level should never go beyond one third of the total prize budget. In other words, the optimal prize-allocation rule should, in principle, give an additional reward to the grand winner of the whole contest.

These results generate immediate implications for optimal reward design in labor competitions involving multi-dimensional indicators. Our results establish that the best use of performance indicators on multiple dimensions (i.e., winning outcomes in different battles) should, in general, involve an indicator of grand overall performance (i.e., the winning outcome of the whole contest), which effectively aggregates all dimensional information. This means that in general all dimensional information should be utilized at least to some extent at an integrated level, instead of being treated completely separately.

Appendix A: Proof of Lemma 2 and Proof of Theorem 1

Proof of Lemma 2

Proof. We consider the prize structures covered by $\mathcal{V}_0$ and $\mathcal{V}_1$. We solve the subgame perfect equilibrium by backward induction. Recall that the ordered pair (n_A, n_B) denotes the number of battles won by the ordered pair of players (A, B).

We first look at the third battle. When $(n_A, n_B) = (2, 0)$, by budget constraint (1), the two players have a common effective prize spread of winning the third battle:

$$v_A(2,0) = v_B(2,0) = v(3) - v(2) = v(1) - v(0) \geq 0.$$

By Lemma 1(i), we have the following equilibrium effort supply

$$x_A(2,0) = x_B(2,0) = \frac{rv_A(2,0)}{4} = \frac{r}{4}(v(1) - v(0)). \tag{a}$$

Each player's winning probability for this battle is

$$p_A(2,0) = p_B(2,0) = \frac{1}{2}. \tag{b}$$

The same results hold when $(n_A, n_B) = (0, 2)$.

When $(n_A, n_B) = (1, 1)$, the two players' common effective prize spread of winning the third battle is

$$v_A(1,1) = v_B(1,1) = v(2) - v(1) \geq 0.$$

By Lemma 1(i), we have the following equilibrium effort supply

$$x_A(1,1) = x_B(1,1) = \frac{rv_A(2,0)}{4} = \frac{r}{4}(v(2) - v(1)). \tag{c}$$

Each player's winning probability for this battle is

$$p_A(1,1) = p_B(1,1) = \frac{1}{2}. \tag{d}$$

We now turn to the second battle. When $(n_A, n_B) = (1, 0)$, the effective prize spread for player A is

$$\begin{aligned} v_A(1,0) &= [p_A(2,0)v(3) + p_B(2,0)v(2) - x_A(2,0)] - [p_A(1,1)v(2) + p_B(1,1)v(1) - x_A(1,1)] \\ &= [\frac{1}{2}v(3) + \frac{1}{2}v(2) - \frac{r}{4}(v(1) - v(0))] - [\frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1))] \\ &= w_A. \end{aligned}$$

Analogously, the effective prize spread for player B is

$$\begin{aligned} v_B(1,0) &= [\frac{1}{2}v(2)+\frac{1}{2}v(1)-\frac{r}{4}(v(2)-v(1))]-[\frac{1}{2}v(1)-\frac{1}{2}v(0)-\frac{r}{4}(v(1)-v(0))] \\ &= w_B. \end{aligned}$$

The prize structures are restricted to $\mathcal{V}_0 \cup \mathcal{V}_1$. Therefore Lemma 1($i$) applies to the second battle, and we have the following equilibrium effort:

$$x_A(1,0)=\frac{rv_A^{r+1}(1,0)v_B^r(1,0)}{[v_A^r(1,0)+v_B^r(1,0)]^2}; x_B(1,0)=\frac{rv_B^{r+1}(1,0)v_A^r(1,0)}{[v_A^r(1,0)+v_B^r(1,0)]^2}. \tag{e}$$

The players' winning probabilities are

$$\begin{aligned} p_A(1,0) &= \frac{x_A^r(1,0)}{x_A^r(1,0)+x_B^r(1,0)}=\frac{v_A^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)}; \\ p_B(1,0) &= \frac{x_B^r(1,0)}{x_A^r(1,0)+x_B^r(1,0)}=\frac{v_B^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)}. \end{aligned} \tag{f}$$

Similarly, when $(n_A, n_B) = (0, 1)$, we have

$$x_A(0,1)=x_B(1,0)=\frac{rv_B^{r+1}(1,0)v_A^r(1,0)}{[v_A^r(1,0)+v_B^r(1,0)]^2};\ x_B(0,1)=x_A(1,0)=\frac{rv_A^{r+1}(1,0)v_B^r(1,0)}{[v_A^r(1,0)+v_B^r(1,0)]^2}. \tag{g}$$

Then players' winning probabilities are

$$p_A(0,1)=p_B(1,0)=\frac{v_B^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)};\ p_B(0,1)=p_A(1,0)=\frac{v_A^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)}. \tag{h}$$

Now we consider the first battle. By (a) to (h), the effective prize spreads are symmetric across the two players:

$$\begin{aligned} &v_A(0,0)=v_B(0,0) \\ &=\{p_A(1,0)[p_A(2,0)v(3)+p_B(2,0)v(2)-x_A(2,0)] \\ &\qquad +p_B(1,0)[p_A(1,1)v(2)+p_B(1,1)v(1)-x_A(1,1)]-x_A(1,0)\} \\ &-\{p_A(0,1)[p_A(1,1)v(2)+p_B(1,1)v(1)-x_A(1,1)] \\ &\qquad +p_B(0,1)[p_A(0,2)v(1)+p_B(0,2)v(0)-x_A(0,2)]-x_A(0,1)\} \\ &=\frac{rv_A^r(1,0)v_B^r(1,0)}{[v_A^r(1,0)+v_B^r(1,0)]^2}[v_B(1,0)-v_A(1,0)]+\frac{v_A^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)}[v(2)-v(0)]. \end{aligned}$$

Applying Lemma 1(i), we have the following equilibrium effort

$$x_A(0,0)=x_B(0,0)=\frac{r}{4}v_A(0,0),$$

and players' winning probabilities are

$$p_A(0,0) = p_B(0,0) = \frac{1}{2}.$$

By (a) to (h), together with the calculation above, the aggregate effort over all three battles equals

$$\begin{aligned} TE_1 &= 2x_A(0,0) + [x_A(1,0) + x_B(1,0)] + p_A(1,0)[x_A(2,0) + x_B(2,0)] + p_B(1,0)[x_A(1,1) + x_B(1,1)] \\ &= \frac{rw_A^r w_B^r}{[w_A^r + w_B^r]^2}[(1-\frac{r}{2})w_A + (1+\frac{r}{2})w_B] + \frac{r}{2}(v(2)-v(1)) + r(v(1)-v(0))\frac{w_A^r}{w_A^r + w_B^r}, \end{aligned}$$

which gives the desired result by incorporating the budget constraints. ■

Proof of Theorem 1

Step 1

Proof. We discuss two cases to show the optimal prize structure must be in $\mathcal{V}_0 \cup \mathcal{V}_1$. In Case 1, $r \in (0, \overline{r}]$; in Case 2, $r \in (\overline{r}, 2]$, where $\overline{r} \approx 1.19$ is the unique solution of $r = 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r$ on $[0, 2]$.

Clearly, a solution $\overline{r}$ exists and must fall in $(1, 2)$. Note that the left side of the equation increases with r and the right side decreases with r when $r \in (1, 2)$. This observation has two implications on $\overline{r}$. First, $\overline{r}$ is unique. Second, a *single-crossing property* holds: $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}} > (r-1)^{\frac{1}{r}}$ if and only if $r < \overline{r}$. This property further yields $\overline{r} < 1.2$, as $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}|_{r=1.2} < (r-1)^{\frac{1}{r}}|_{r=1.2}$. Therefore, we have $\overline{r} \in (1, 1.2)$. More specifically, one can obtain the numerical solution of $\overline{r} \approx 1.1935$. This property will be employed when we prove Properties 1-3.

In Case 1, we show that $\mathcal{V}_2 \cup \mathcal{V}_3 = \emptyset$ in Lemma 3 as follow. As for Case 2, we show that the effort-maximizing prize structure must be within $\mathcal{V}_1$ by proving the following Lemmas 4-5 and Propositions 1-3. ■

LEMMA 3 *(i) For any $r \in (0, \overline{r}]$, we have $\mathcal{V}_0 \neq \emptyset$, $\mathcal{V}_1 \neq \emptyset$, and $\mathcal{V}_2 = \mathcal{V}_3 = \emptyset$. Thus $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$;*
(ii) For any $r \in (\overline{r}, 2]$, we have $\mathcal{V}_i \neq \emptyset, \forall i$.

Proof. It is clear that $\mathcal{V}_0 \neq \emptyset$ and $\mathcal{V}_1 \neq \emptyset$ in both cases, since $w_A = w_B$ when $(v(0), v(1), v(2), v(3)) = (0, \frac{1}{3}, \frac{2}{3}, 1)$. For the first part, by Definition 3, it suffices to show that w_A and w_B satisfy either $r \leq 1 + (\frac{w_A}{w_B})^r$ when $w_A \leq w_B$ or $r \leq 1 + (\frac{w_B}{w_A})^r$ when $w_A \geq w_B$ for any given $r \in (0, \overline{r}]$. By the definitions of w_A and w_B, we can easily verify that $(\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r \leq (\frac{w_A}{w_B})^r$ when $w_A \leq w_B$; and $(\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r \leq (\frac{w_B}{w_A})^r$ when $w_A \geq w_B$ for any $r > 0$. By the definition of $\overline{r}$ and the fact that $r \in (0, \overline{r}]$, we have $r \leq 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r$. Therefore $r \leq 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r \leq 1 + (\frac{w_A}{w_B})^r$; and $r \leq 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r \leq 1 + (\frac{w_B}{w_A})^r$, $\forall r \in (0, \overline{r}]$.

For the second part, we can verify that the feasible prize structure $(v(0), v(1), v(2), v(3)) = (0, 0, 1, 1)$ belongs to $\mathcal{V}_2$, $\forall r \in (\overline{r}, 2]$. To show this, under that prize structure, we have $w_A =$

$\frac{1}{2} + \frac{r}{4} > w_B = \frac{1}{2} - \frac{r}{4}$. And $1 + (\frac{w_B}{w_A})^r = 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r < r$ holds when $r \in (\overline{r}, 2]$, which follows from the definition of $\overline{r}$. Therefore $\mathcal{V}_2 \neq \emptyset$. In addition, note that the feasible prize structure $(v(0), v(1), v(2), v(3)) = (0, \frac{1}{2}, \frac{1}{2}, 1)$ belongs to $\mathcal{V}_3$ for any $r \in (\overline{r}, 2]$, since the resulting $w_A = \frac{1}{2}(\frac{1}{2} - \frac{r}{4}) < w_B = \frac{1}{2}(\frac{1}{2} + \frac{r}{4})$ and $1 + (\frac{w_A}{w_B})^r = 1 + (\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}})^r < r$ holds when $r \in (\overline{r}, 2]$. Hence $\mathcal{V}_3 \neq \emptyset$. ■

In Case 2 where $r \in (\overline{r}, 2]$, $\mathcal{V}_2 \cup \mathcal{V}_3$ are no longer empty, we have to compare the optimal prize structures across $\mathcal{V}_i$. Note that, for any prize structure in $\mathcal{V}_2 \cup \mathcal{V}_3$, players' equilibrium effort in the second battle of the contest is calculated by using Lemma 1(ii), instead of using Lemma 1(i) as we calculate TE_1.

We continue to prove our argument for Case 2. We first show that any prize structure in $\mathcal{V}_0 \backslash \mathcal{V}_1$ is not optimal in Lemma 4. Second, we calculate the expected aggregate effort TE_2 and TE_3, for the prize structures in $\mathcal{V}_2$ and $\mathcal{V}_3$ respectively, and we find $\sup_{\mathcal{V}_2} TE_2$, $\sup_{\mathcal{V}_3} TE_3$. We present the results in Proposition 1 and Proposition 2. Third, we show that $\sup_{\mathcal{V}_2} TE_2$, $\sup_{\mathcal{V}_3} TE_3$ are two lower bounds for $\max_{\mathcal{V}_1} TE_1$ in Lemma 5. Combining the results, we conclude that $\max_{\mathcal{V}_1} TE_1 \geq \max\{sup_{\mathcal{V}_2} TE_2, sup_{\mathcal{V}_3} TE_3\}$ in Proposition 3. We then only need to restrict our search within $\mathcal{V}_1$ to determine the optimal prize structure when $r \in (\overline{r}, 2]$.

Lemma 4 $\forall r \in (\overline{r}, 2]$, *any prize structure in* $\mathcal{V}_0 \backslash \mathcal{V}_1$ *is strictly dominated by a prize structure in* $\mathcal{V}_0 \cap \mathcal{V}_1$.

Proof. For any prize profile in $\mathcal{V}_0$ or $\mathcal{V}_1$, the resulting aggregate effort is denoted by TE_1 in (4). Take any prize profile in $\mathcal{V}_0 \backslash \mathcal{V}_1$. We construct a prize structure in $\mathcal{V}_0 \cap \mathcal{V}_1$ that induces a strictly higher level of effort than the given prize structure in $\mathcal{V}_0 \backslash \mathcal{V}_1$.

By Lemma 2 and the two formulas $v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}$, $v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}$, we have

$$TE_1 = \frac{r w_A^r w_B^r}{[w_A^r + w_B^r]^2}[(1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1))] + \frac{r}{2}(v(2) - v(1)) + r(v(1) - v(0))\frac{w_A^r}{w_A^r + w_B^r},$$

for prize structures in $\mathcal{V}_0 \cup \mathcal{V}_1$ when $r \in (\overline{r}, 2]$.

Recall that any prize structure $\{v(0), v(1), v(2), v(3)\} \in \mathcal{V}_0 \backslash \mathcal{V}_1$ satisfies $w_B > w_A$, so that $v(1) - v(0) > v(2) - v(1)$ which is equivalent to $v(2) \in [\frac{1}{2}, \frac{2-v(0)}{3})$.

Construct a new prize structure $\{\widetilde{v}(0), \widetilde{v}(1), \widetilde{v}(2), \widetilde{v}(3)\} = \{v(0), \frac{1+v(0)}{3}, \frac{2-v(0)}{3}, v(3)\} \in \mathcal{V}_0 \cap \mathcal{V}_1$. We shall show the aggregate effort induced by the new prize structure $\{\widetilde{v}(0), \widetilde{v}(1), \widetilde{v}(2), \widetilde{v}(3)\}$ is higher than the aggregate effort induced by the given prize structure $\{v(0), v(1), v(2), v(3)\}$. And it is sufficient to prove that $\frac{d}{dv(2)} TE_1 > 0$ when $w_B > w_A$ and $r \in (\overline{r}, 2]$. For that, we first take derivative of TE_1 with respect to $v(2)$:

$$\begin{aligned}\frac{d}{dv(2)}TE_1 &= \frac{d}{dv(2)}\{\frac{rw_A^r w_B^r}{[w_A^r+w_B^r]^2}\}[(1+\frac{r^2}{4})(v(1)-v(0))+(1-\frac{r^2}{4})(v(2)-v(1))] \\ &+ \frac{rw_A^r w_B^r}{[w_A^r+w_B^r]^2}\frac{d}{dv(2)}\{(1+\frac{r^2}{4})(v(1)-v(0))+(1-\frac{r^2}{4})(v(2)-v(1))\} \\ &+ \frac{d}{dv(2)}\{\frac{r}{2}(v(2)-v(1))\}+\frac{d}{dv(2)}\{r(v(1)-v(0))\}\frac{w_A^r}{w_A^r+w_B^r} \\ &+ r(v(1)-v(0))\frac{d}{dv(2)}\{\frac{w_A^r}{w_A^r+w_B^r}\}.\end{aligned}$$

It follows from $w_B > w_A$ and the definitions of w_A, w_B that $\frac{d}{dv(2)}\{\frac{w_A^r}{w_A^r+w_B^r}\} > 0$ and $\frac{d}{dv(2)}\{\frac{w_A^r w_B^r}{[w_A^r+w_B^r]^2}\} = \frac{d}{dv(2)}\{\frac{w_A^r}{w_A^r+w_B^r}[1-\frac{w_A^r}{w_A^r+w_B^r}]\} > 0$.

Hence

$$\begin{aligned}\frac{d}{dv(2)}TE_1 &> \frac{rw_A^r w_B^r}{[w_A^r+w_B^r]^2}\frac{d}{dv(2)}\{(1+\frac{r^2}{4})(1-v(2)-v(0))+(1-\frac{r^2}{4})(2v(2)-1)\} \\ &+ \frac{d}{dv(2)}\{\frac{r}{2}(2v(2)-1)\}+\frac{d}{dv(2)}\{r(1-v(2)-v(0))\}\frac{w_A^r}{w_A^r+w_B^r} \\ &= \frac{rw_A^r w_B^r}{[w_A^r+w_B^r]^2}[(1+\frac{r^2}{4})\times(-1)+(1-\frac{r^2}{4})\times 2]+r-r\frac{w_A^r}{w_A^r+w_B^r}.\end{aligned}$$

Note that $[(1+\frac{r^2}{4})\times(-1)+(1-\frac{r^2}{4})\times 2] = 1-\frac{3}{4}r^2 < 0$ when $r \in (\overline{r}, 2]$. In addition, $w_B > w_A$ implies $\frac{w_A^r}{w_A^r+w_B^r} < \frac{1}{4}$. Thus, when $r \in (\overline{r}, 2]$, we have:

$$\begin{aligned}\frac{d}{dv(2)}TE_1 &> \frac{r}{4}[1-\frac{3}{4}r^2]+r-\frac{r}{2} \\ &= \frac{3}{4}r-\frac{3}{16}r^3 \\ &\geq 0.\end{aligned}$$

In sum, $\frac{d}{dv(2)}TE_1 > 0$ when $w_B > w_A$ and $r \in (\overline{r}, 2]$. We thus have that any structure in $\mathcal{V}_0 \backslash \mathcal{V}_1$ with $v(2) \in [\frac{1}{2}, \frac{2-v(0)}{3})$ is dominated by $\{\widetilde{v}(0), \widetilde{v}(1), \widetilde{v}(2), \widetilde{v}(3)\} = \{v(0), \frac{1+v(0)}{3}, \frac{2-v(0)}{3}, v(3)\}$. ■

Since Lemma 4 establishes that any prize structure inside $\mathcal{V}_0$ is dominated by a prize structure in $\mathcal{V}_1$, we can ignore all prize structures in $\mathcal{V}_0 \backslash \mathcal{V}_1$ to identify the optimal prize structure. Note that any prize structure within $\mathcal{V}_1$ satisfies $w_A \geq w_B$, i.e., the winner of the first battle has a higher prize spread.

We next pin down the expected aggregate effort induced by prize profiles in $\mathcal{V}_2$ and $\mathcal{V}_3$, respectively. For any given prize structure, computing aggregate effort requires solving the players' choice of effort on each stage game for every possible path. For any prize structure in $\mathcal{V}_2 \cup \mathcal{V}_3$, players' equilibrium effort in the second battle of the contest is calculated by using Lemma 1(ii). The uniqueness of subgame perfect equilibrium is discussed after Lemma 1.

In the next two propositions, we show that we can ignore the prize profiles in $\mathcal{V}_2$ and $\mathcal{V}_3$ when searching for the optimal prize structure. As mentioned, the idea is to show that the highest possible expected aggregate effort generated by prizes in either $\mathcal{V}_2$ or $\mathcal{V}_3$ is dominated by at least one of the two bounds in Lemma 5.

PROPOSITION 1 $\forall r \in (\overline{r}, 2]$, *a prize structure in* $\mathcal{V}_2$ *induces the aggregate effort*

$$TE_2 = \frac{r}{2}(1-2v(0)) + (1-\frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}(2-r)[\frac{r}{2} - (\frac{1}{2}+\frac{r}{4})v(0) + (\frac{1}{2}-\frac{3r}{4})v(2)]. \quad (5)$$

And we have $\sup_{\mathcal{V}_2} TE_2 = TE_2^* := \frac{r}{2} + \frac{(r-1)(2-r)}{(1+\frac{3r}{2})(r-1)^{\frac{1}{r}}+\frac{3r}{2}-1}$, *which is lower than* $\max_{\mathcal{V}_1} TE_1$. *Moreover, there exists no prize structure in* $\mathcal{V}_2$ *that can induce* $\sup TE_2$.

Proof. We solve the aggregate effort TE_2 and then maximize TE_2 in (5) within $\mathcal{V}_2$. We note that TE_2 decreases in $v(2)$ and $v(2) > \frac{r[1+(r-1)^{\frac{1}{r}}]}{[(r-1)^{\frac{1}{r}}(1+\frac{3}{2}r)+\frac{3}{2}r-1]}$ within $\mathcal{V}_2$. By calculations, the proposition follows.

We first solve the game by backward induction and calculate players' expected aggregate effort. Recall (n_A, n_B) denotes the history of the game. We start from the third battle, which can be solved in the same way as Lemma 2.

When $(n_A, n_B) = (2, 0)$ or $(0, 2)$

$$x_A(2,0) = x_B(2,0) = x_A(0,2) = x_B(0,2) = \frac{rv_A(2,0)}{4} = \frac{r}{4}(v(1)-v(0)).$$

The winning probabilities are

$$p_A(2,0) = p_B(2,0) = p_A(0,2) = p_B(0,2) = \frac{1}{2}.$$

When $(n_A, n_B) = (1, 1)$, we have

$$x_A(1,1) = x_B(1,1) = \frac{rv_A(1,1)}{4} = \frac{r}{4}(v(2)-v(1)).$$

The winning probabilities are

$$p_A(2,0) = p_B(2,0) = \frac{1}{2}.$$

Next, we look at the second battle. When $(n_A, n_B) = (1, 0)$, recall from the proof of Lemma 2 that the effective prize spreads are as follows:

$$v_A(1,0) = \frac{1}{2}(1-v(1)-v(0)) + \frac{r}{4}(1-3v(1)+v(0)) = w_A,$$

$$v_B(1,0) = \frac{1}{2}(1-v(1)-v(0)) + \frac{r}{4}(-1+3v(1)-v(0)) = w_B.$$

$v_A(1,0) \geq v_B(1,0)$ is equivalent to $w_A \geq w_B$. As the prize profiles are in $\mathcal{V}_2$, applying Lemma

1(ii) gives the equilibrium effort:

$$x_A(1,0) = (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})v_B(1,0),$$

$$\widetilde{x_B}(1,0) = \begin{cases} (1-\frac{1}{r})v_B(1,0), & \text{with probability } q = \frac{v_B(1,0)}{v_A(1,0)}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \text{with probability } 1-q. \end{cases}$$

The winning probabilities are

$$p_A(1,0) = 1-(1-\frac{1}{r})q;\ p_B(1,0) = (1-\frac{1}{r})q.$$

Similarly, when $(n_A, n_B) = (0,1)$, we have

$$\widetilde{x_A}(0,1) = \widetilde{x_B}(1,0) = \begin{cases} (1-\frac{1}{r})v_B(1,0), & \text{with probability } q = \frac{v_B(1,0)}{v_A(1,0)}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \text{with probability } 1-q. \end{cases}$$

$$x_B(0,1) = x_A(1,0) = (\frac{1}{r-1})^{\frac{1}{r}}(1-\frac{1}{r})v_B(1,0).$$

The winning probabilities are

$$p_A(0,1) = (1-\frac{1}{r})q;\ p_B(0,1) = 1-(1-\frac{1}{r})q.$$

We have

$$E[\widetilde{x_A}(0,1)] = E[\widetilde{x_B}(1,0)] = q(1-\frac{1}{r})v_B(1,0) = (1-\frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}\frac{v_B^2(1,0)}{v_A(1,0)}.$$

Now we come to the first battle. The common prize spread is

$$\begin{aligned} & v_A(0,0) = v_B(0,0) \\ & = \{p_A(1,0)[p_A(2,0)v(3) + p_B(2,0)v(2) - x_A(2,0)] \\ & \qquad + p_B(1,0)[p_A(1,1)v(2) + p_B(1,1)v(1) - x_A(1,1)] - x_A(1,0)\} \\ & - \{p_A(0,1)[p_A(1,1)v(2) + p_B(1,1)v(1) - x_A(1,1)] \\ & \qquad + p_B(0,1)[p_A(0,2)v(1) + p_B(0,2)v(0) - x_A(0,2)] - E[\widetilde{x_A}(0,1)]\} \\ & = [1-(1-\frac{1}{r})q][v(2)-v(0)] + (1-\frac{1}{r})v_B(1,0)[q-(\frac{1}{r-1})^{\frac{1}{r}}]. \end{aligned}$$

Thus the effort supply is

$$x_A(0,0) = x_B(0,0) = \frac{r}{4}v_A(0,0),$$

and winning probabilities are

$$p_A(0,0) = p_B(0,0) = \frac{1}{2}.$$

Aggregating over the three battle, we have the aggregate effort:

$$\begin{aligned}TE_2 &= 2x_A(0,0) + x_A(1,0) + E[\widetilde{x_B}(1,0)] \\ &\quad + p_A(1,0)(x_A(2,0) + x_B(2,0)) + p_B(1,0)(x_A(1,1) + x_B(1,1)) \\ &= \frac{r}{2}(1 - 2v(0)) + (1 - \frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}(2-r)[\frac{r}{2} - (\frac{1}{2} + \frac{r}{4})v(0) + (\frac{1}{2} - \frac{3r}{4})v(2)].\end{aligned}$$

Given that, we now maximize effort TE_2 subject to prize profiles are in $\mathcal{V}_2$. By definition of $\mathcal{V}_2$, $w_A \geq w_B$ and $1 + (\frac{w_B}{w_A})^r < r \leq 2$ can be simplified as $(r-1)^{\frac{1}{r}} w_A > w_B$, which implies the following inequality involving $v(2)$ and $v(0)$:

$$\begin{aligned}v(2) &> \frac{(\frac{1}{2}(r-1)^{\frac{1}{r}} - \frac{1}{4}r(r-1)^{\frac{1}{r}} - \frac{1}{2} - \frac{1}{4}r)v(0) + \frac{1}{2}r + \frac{1}{2}r(r-1)^{\frac{1}{r}}}{\frac{3}{4}r(r-1)^{\frac{1}{r}} - \frac{1}{2} + \frac{1}{2}(r-1)^{\frac{1}{r}} + \frac{3}{4}r} \\ &= \frac{(1 - \frac{1}{2}r)(r-1)^{\frac{1}{r}} - (1 + \frac{1}{2}r)}{[(1 + \frac{3}{2}r)(r-1)^{\frac{1}{r}} + \frac{3}{2}r - 1]}v(0) + \frac{r[1 + (r-1)^{\frac{1}{r}}]}{[(1 + \frac{3}{2}r)(r-1)^{\frac{1}{r}} + \frac{3}{2}r - 1]}.\end{aligned}$$

Clearly, TE_2 is decreasing in $v(2)$ and $v(0)$ when $r \in (\overline{r}, 2]$. However optimal $v(0) = 0$ does not follow immediately. This is because we have to take care of the role of $v(0)$ in the lower bound of $v(2)$ above. After the calculations, we can show for any prize profile in $\mathcal{V}_2$, the effort induced is lower than $TE_2(v(0) = 0, v(2) = \frac{r[1+(r-1)^{\frac{1}{r}}]}{[(1+\frac{3}{2}r)(r-1)^{\frac{1}{r}} + \frac{3}{2}r - 1]})$, which is the following. The full arguments and calculation are in the online appendix.

$$\begin{aligned}TE_2(v(0) = 0, v(2) &= \frac{r[1 + (r-1)^{\frac{1}{r}}]}{[(r-1)^{\frac{1}{r}}(1 + \frac{3}{2}r) + \frac{3}{2}r - 1]}) \\ &= \frac{r}{2} + \frac{(r-1)(2-r)}{[(r-1)^{\frac{1}{r}}(1 + \frac{3r}{2}) + \frac{3r}{2} - 1]}.\end{aligned}$$

■

Analogous to Proposition 1, we provide the aggregate effort induced by prize structures in $\mathcal{V}_3$, and establish an upper bound that is not attainable in $\mathcal{V}_3$.

PROPOSITION 2 $\forall r \in (\overline{r}, 2]$, *a prize structure in* $\mathcal{V}_3$ *induces the aggregate effort*

$$TE_3 = \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}(2+r)[-\frac{r}{2} - (\frac{1}{2} - \frac{r}{4})v(0) + (\frac{1}{2} + \frac{3}{4}r)v(2)]. \quad (6)$$

And we have $\sup_{\mathcal{V}_3} TE_3 = TE_3^* := \frac{(\frac{r^2}{4} + \frac{r}{2})(r-1)^{\frac{1}{r}} + \frac{5}{4}r^2 + \frac{r}{2} - 2}{(\frac{3r}{2} - 1)(r-1)^{\frac{1}{r}} + \frac{3r}{2} + 1}$, *which is lower than* $\max_{\mathcal{V}_1} TE_1$. *Moreover, there exists no prize structure in* $\mathcal{V}_3$ *that can induce* $\sup TE_3$.

Proof. As before, we calculate the aggregate effort TE_3 and maximize TE_3 in (6) within $\mathcal{V}_3$.

The third battle can be analyzed identically as in Proposition 1. We now look at the second battle. Recall (n_A, n_B) denotes the history of the contest. When $(n_A, n_B) = (1, 0)$ the effective prize spreads are respectively

$$v_A(1,0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(1 - 3v(1) + v(0)) = w_A,$$

$$v_B(1,0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(-1 + 3v(1) - v(0)) = w_B.$$

$v_A(1,0) \leq v_B(1,0)$ is equivalent to $w_A \leq w_B$. As the prize profiles are in $\mathcal{V}_3$, applying Lemma 1(ii) gives the equilibrium effort:

$$\widetilde{x_A}(1,0) = \begin{cases} (1 - \frac{1}{r})v_A(1,0), & \text{with probability } q = \frac{v_A(1,0)}{v_B(1,0)}(\frac{1}{r-1})^{\frac{1}{r}}, \\ 0, & \text{with probability } 1 - q. \end{cases}$$

$$x_B(1,0) = (\frac{1}{r-1})^{\frac{1}{r}}(1 - \frac{1}{r})v_A(1,0).$$

The winning probability are

$$p_A(1,0) = (1 - \frac{1}{r})q, \ p_B(1,0) = 1 - (1 - \frac{1}{r})q.$$

$(n_A, n_B) = (0,1)$ is dual case of $(n_A, n_B) = (1,0)$.

Now we come to the first battle where $(n_A, n_B) = (0,0)$. We pin down the common effective prize spread:

$$v_A(0,0) = v_B(0,0) = (1 - \frac{1}{r})q(v(2) - v(0)) + (1 - \frac{1}{r})v_A(1,0)[\frac{1}{r-1})^{\frac{1}{r}} - q].$$

Thus we have the effort supply

$$x_A(0,0) = x_B(0,0) = \frac{r}{4}v_A(0,0),$$

and the winning probabilities

$$p_A(0,0) = p_B(0,0) = \frac{1}{2}.$$

Aggregate effort thus equals

$$\begin{aligned} TE_3 &= 2x_A(0,0) + E[\widetilde{x_A}(1,0)] + x_B(1,0) \\ &\quad + p_A(1,0)(x_A(2,0) + x_B(2,0)) + p_B(1,0)(x_A(1,1) + x_B(1,1)) \\ &= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r-1})^{\frac{1}{r}}(2 + r)[-\frac{r}{2} - (\frac{1}{2} - \frac{r}{4})v(0) + (\frac{1}{2} + \frac{3}{4}r)v(2)]. \end{aligned}$$

Within $\mathcal{V}_3$, $v(2) < \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2}-1)+(\frac{3r}{2}+1)} + \frac{-(r-1)^{\frac{1}{r}}(\frac{1}{2}+\frac{r}{4})+\frac{1}{2}-\frac{r}{4}}{(r-1)^{\frac{1}{r}}(\frac{3r}{2}-1)+(\frac{3r}{2}+1)}v(0)$ when $r \in (\overline{r}, 2]$. And when $r \in (\overline{r}, 2]$, we have $\frac{-(r-1)^{\frac{1}{r}}(\frac{1}{2}+\frac{r}{4})+\frac{1}{2}-\frac{r}{4}}{(r-1)^{\frac{1}{r}}(\frac{3r}{2}-1)+(\frac{3r}{2}+1)} < 0$. In addition, TE_3 is decreasing in $v(0)$ and increasing in $v(2)$ when $r \in (\overline{r}, 2]$. We thus have for any prize profile in $\mathcal{V}_3$, the resulting aggregate effort is lower than $TE_3(v(0)=0, v(2)=\frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2}-1)+(\frac{3r}{2}+1)})$, which is the following.

$$
\begin{aligned}
TE_3(v(0)=0, v(2) &= \frac{r(1+(r-1)^{\frac{1}{r}})}{(r-1)^{\frac{1}{r}}(\frac{3r}{2}-1)+(\frac{3r}{2}+1)}) \\
&= \frac{(\frac{r^2}{4}+\frac{r}{2})(r-1)^{\frac{1}{r}}+\frac{5}{4}r^2+\frac{r}{2}-2}{[1+\frac{3r}{2}+(\frac{3r}{2}-1)(r-1)^{\frac{1}{r}}]}.
\end{aligned}
$$

We leave the full calculation details in the online appendix. ■

We successively show that the highest possible expected aggregate effort generated by prize structures in either $\mathcal{V}_2$ or $\mathcal{V}_3$ is dominated by at least one of these two bounds. This fact enables us to ignore $\mathcal{V}_2$ and $\mathcal{V}_3$ when we search for the optimum.

Lemma 5 $\forall r \in (\overline{r}, 2]$, *we have* $\max_{\mathcal{V}_1} TE_1 \geq \max\{\frac{r}{2}+\frac{(r-1)(2-r)}{(\frac{3r}{2}-1)+(1+\frac{3r}{2})(r-1)^{\frac{1}{r}}}, \frac{\frac{5}{4}r^2+\frac{r}{2}-2+(\frac{r^2}{4}+\frac{r}{2})(r-1)^{\frac{1}{r}}}{(\frac{3r}{2}-1)(r-1)^{\frac{1}{r}}+\frac{3r}{2}+1}\}$.

Proof. Pick two prize structures $(v(0)=0, v(2)=\frac{2r[(r-1)^{\frac{1}{r}}+1]}{3r+3r(r-1)^{\frac{1}{r}}+2(r-1)^{\frac{1}{r}}-2}) \in \mathcal{V}_1$ and $(v(0)=0, v(2)=\frac{2r[(r-1)^{\frac{1}{r}}+1]}{3r+3r(r-1)^{\frac{1}{r}}-2(r-1)^{\frac{1}{r}}+2}) \in \mathcal{V}_0$. The two resulting aggregate effort levels generated by the two prize structures provide two lower bounds for the maximal effort inducible in $\mathcal{V}_1$. The remaining substitutions and calculations are included in the online appendix. ■

Proposition 3 $\forall r \in (\overline{r}, 2]$, *the optimal prize structure cannot be in* $\mathcal{V}\backslash\mathcal{V}_1$ *and must be in* $\mathcal{V}_1$ *if it exists.*

Proof. According to Lemma 5, Propositions 1 and 2, we have $\max_{\mathcal{V}_1} TE_1 \geq \max\{sup_{\mathcal{V}_2}TE_2, sup_{\mathcal{V}_3}TE_3\}$. Together with Lemma 4, Proposition 3 follows. ■

Step 2

Proof. It suffices to show that $v^*(3)=1$ under a release restriction of $v(0)+v(3)\leq 1$, assuming that $v^*(0)=0$, since only prize spread will affect a player' incentive. For any prize structure $(v(0), v(1), v(2), v(3))$ with $v(0)=0$ and $v(3)<1$, we construct a new prize structure $(0, \frac{v(1)}{v(3)}, \frac{v(2)}{v(3)}, 1)$ such that the difference $v(1)-v(0)$, $v(2)-v(1)$, $v(3)-v(2)$ increase proportionally by a factor of $\frac{1}{v(3)}(>1)$, and therefore both players would exert larger effort at all reachable states (n_A, n_B) by a factor of $\frac{1}{v(3)}$, and each player wins with the same winning probability at each reachable state (n_A, n_B) as before. As a result, this new prize structure $(0, \frac{v(1)}{v(3)}, \frac{v(2)}{v(3)}, 1)$ induces larger expected aggregate effort than the given prize structure $(v(0), v(1), v(2), v(3))$.

Now, we differentiate $\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}$ with respect to $v(2)$ to identify the optimal $v(2)$:

$$\begin{aligned}
&\frac{d}{dv(2)}[\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}] \\
= \ & (1-2\frac{w_A^r}{w_A^r+w_B^r})\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})[(1+\frac{r^2}{4})v(1)+(1-\frac{r^2}{4})(v(2)-v(1))] \\
&+\frac{w_A^r}{w_A^r+w_B^r}\frac{w_B^r}{w_A^r+w_B^r}[(1+\frac{r^2}{4})(-1)+(1-\frac{r^2}{4})2]+1+\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})v(1)+\frac{w_A^r}{w_A^r+w_B^r}(-1) \\
= \ & \frac{w_B^r-w_A^r}{w_A^r+w_B^r}[(1+\frac{r^2}{4})v(1)+(1-\frac{r^2}{4})(v(2)-v(1))]\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r}) \\
&+\frac{w_A^r}{w_A^r+w_B^r}\frac{w_B^r}{w_A^r+w_B^r}(1-\frac{3r^2}{4})+1+\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})v(1)-\frac{w_A^r}{w_A^r+w_B^r}.
\end{aligned}$$

We first argue that the optimal $v(2)\notin[\frac{1}{2},\frac{2}{3})$. Suppose that $v(2)\in[\frac{1}{2},\frac{2}{3})$, i.e. $w_B>w_A$. w_A and w_B are non-negative because of the monotonicity of prizes. Note that $I\equiv\frac{w_B^r-w_A^r}{w_A^r+w_B^r}[(1+\frac{r^2}{4})v(1)+(1-\frac{r^2}{4})(v(2)-v(1))]\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})\geq 0$ because $\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})\geq 0$, $II\equiv\frac{w_A^r}{w_A^r+w_B^r}\frac{w_B^r}{w_A^r+w_B^r}(1-\frac{3r^2}{4})>-\frac{1}{2}$, $III\equiv 1$, $IV\equiv\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})v(1)\geq 0$, and $V\equiv\frac{w_A^r}{w_A^r+w_B^r}<\frac{1}{2}$ because $w_B>w_A$. Thus, $\frac{d}{dv(2)}[\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}]>0$ when $v(2)\in[0,\frac{2}{3})$. As a result, $TE_1(v(0),v(2))|_{v(0)=0}<TE_1(v(0),v(2))|_{v(0)=0,v(2)=\frac{2}{3}}$ for any $v(2)\in[\frac{1}{2},\frac{2}{3})$. We now turn to consider $\frac{d}{dv(2)}[\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}]$ for the remaining case where $w_B\leq w_A$, i.e. $v(2)\in[\frac{2}{3},1]$ to pin down the optimal $v(2)$. Recall $\eta\equiv\frac{w_B}{w_A}$ can be viewed as a function of $v(0)$, $v(2)$ and r. One can verify that $\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r+w_B^r})=-r\frac{(\frac{w_B}{w_A})^{r-1}}{[1+(\frac{w_B}{w_A})]^2}\frac{1}{w_A^2}[\frac{w_A-w_B}{2}-\frac{3r}{4}(w_A+w_B)]$. Recall $v(1)-v(0)=\frac{w_A+w_B}{2}-\frac{w_A-w_B}{r}$ and $v(2)-v(1)=\frac{w_A+w_B}{2}+\frac{w_A-w_B}{r}$. Thus $\frac{[(1+\frac{r^2}{4})(v(1)-v(0))+(1-\frac{r^2}{4})(v(2)-v(1))]}{w_A}=[(1-\frac{r}{2})+(1+\frac{r}{2})\eta]$. Using notation η, we have $I=\frac{\eta^r-1}{1+\eta^r}\frac{r}{2}\frac{\eta^{r-1}}{[1+\eta^r]^2}[(\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta][(1-\frac{r}{2})+(1+\frac{r}{2})\eta]$, $IV=\frac{r}{2}\frac{\eta^{r-1}}{[1+\eta^r]^2}[(\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta][(\frac{1}{2}-\frac{1}{r})+(\frac{1}{2}+\frac{1}{r})\eta]$, and $II+III+V=\frac{\eta^r}{[1+\eta^r]^2}[2-\frac{3r^2}{4}+\eta^r]$. After substitution and rearrangement, we have

$$\begin{aligned}
&\frac{d}{dv(2)}[\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}] \\
&=\{\frac{\eta^{r-1}}{[1+\eta^r]^2}\{\frac{r}{2}[(\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta][\frac{-1+\eta^r}{1+\eta^r}((1-\frac{r}{2})+(1+\frac{r}{2})\eta)+(\frac{1}{2}-\frac{1}{r})+(\frac{1}{2}+\frac{1}{r})\eta] \\
&\qquad\qquad +\eta[2-\frac{3r^2}{4}+\eta^r]\}\}|_{v(0)=0} \\
&=\{\frac{\eta^{r-1}}{[1+\eta^r]^2}\frac{1}{2[1+\eta^r]}\{[(\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta][(1-\frac{r}{2})(-r-1+(r-1)\eta^r) \\
&\qquad\qquad +(1+\frac{r}{2})(-r+1+(r+1)\eta^r)\eta]+2\eta(1+\eta^r)(2-\frac{3r^2}{4}+\eta^r)\}\}|_{v(0)=0} \\
&=\{\frac{\eta^{r-1}}{[1+\eta^r]^2}\frac{1}{2[1+\eta^r]}D(\eta,r)\}|_{v(0)=0}.
\end{aligned}$$

Therefore, $\frac{d}{dv(2)}[\frac{TE_1(v(0),v(2))}{r}|_{v(0)=0}]\overset{sign}{=}D(\eta(v(0)=0,v(2),r),r)$ whenever $\eta(v(0)=0,v(2),r)$

is positive. And $\eta(v(0)=0, v(2), r) = \frac{w_B}{w_A}|_{v(0)=0} = \frac{(\frac{1}{2}+\frac{r}{4})(1-v(2))+(\frac{1}{2}-\frac{r}{4})(2v(2)-1)}{(\frac{1}{2}-\frac{r}{4})(1-v(2))+(\frac{1}{2}+\frac{r}{4})(2v(2)-1)} > 0$ holds whenever $v(2) < 1$. ■

Step 3

Proof. The existence and uniqueness of $\underline{r}$ is revealed by the following Property 1. Proofs for Properties 1-3 are relegated to the online appendix.

Property 1 $D(\eta, r)|_{\eta=\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}}$ *strictly decreases with* $r \in (0, \overline{r}]$ *and has a unique root of* $\underline{r} \approx 1.0884$ *in this range.*

By the single-crossing property of $\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}$ and $(r-1)^{\frac{1}{r}}$ that is established in the discussion following Definition 4, $\underline{\eta}_r$ in the following Property 2 is well defined.

Property 2 $D(\eta, r)|_{\eta=(r-1)^{\frac{1}{r}}}$ *strictly increases with* $r \in (\overline{r}, 2]$ *and has a unique root of* $r^* \approx 1.31$ *in this range.*

Property 3 $D(\eta, r)$ *strictly increases with* η *when* $\eta \geq \underline{\eta}_r$, $\forall r \in [0, 2]$, *where*

$$\underline{\eta}_r := \max\{\frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, (r-1)^{\frac{1}{r}}\} = \begin{cases} \frac{\frac{1}{2}-\frac{r}{4}}{\frac{1}{2}+\frac{r}{4}}, & \text{if } r \in (0, \overline{r}], \\ (r-1)^{\frac{1}{r}}, & \text{if } r \in (\overline{r}, 2]. \end{cases}$$ [27]

$\underline{\eta}_r$ provides an applicable lower bound for the prize-spread ratio (i.e. the lower bound of η) in battle 2 for $r \in (0, 2]$. ■

[27] Note that $(0, \overline{r}] \subset (0, 1.2]$.

Appendix B: Proof of Theorem 2

Proof. In Lemmas 6 and 7, we will provide the expected aggregate effort under the prize structures in $\mathcal{V}_4$ and $\mathcal{V}_5$ and characterize the optimal prizes in $\mathcal{V}_4$ and $\mathcal{V}_5$, respectively. Combining both lemmas, we can identify the optimal prize structures for each $r > 2$. For a given prize structure, computing players' aggregate effort requires solving the players' choice of effort on each stage game for every possible path. Since Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996) verify the existence of a unique mixed equilibrium described by Lemma 1(iii) in a one-shot all-pay contest, the corresponding equilibrium is unique for each stage game, and the usual backwards induction argument shows that the subgame perfect equilibrium of the whole contest is unique. ■

LEMMA 6 *When $r > 2$, the expected aggregate effort equals $TE_4 = 1 - 2v(0)$ for all prize profiles in $\mathcal{V}_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\}$. The maximum aggregate effort in $\mathcal{V}_4$ equals* 1*, which is achieved by any prize structure* $(0, 1 - v(2), v(2), 1)$*,* $\forall v(2) \in [\frac{2}{3}, 1]$.

Proof. Note that when $r > 2$, Lemma 1(iii) applies. As usual, we solve the game by backward induction.

First, consider battle 3. Given history $(n_A, n_B) = (2, 0)$, the common effective prize spread is:

$$v_A(2,0) = v_B(2,0) = v(1) - v(0) \geq 0.$$

Thus expected effort supply is given by

$$E[\widetilde{x_A}(2,0)] = \frac{v(1) - v(0)}{2},$$

$$E[\widetilde{x_B}(2,0)] = \frac{v(1) - v(0)}{2}.$$

where $G^i_{(n_A, n_B)}(\cdot)$, $i = A, B$ denotes the cumulative distribution function of player i. e

The winning probabilities are

$$p_A(2,0) = p_B(2,0) = \frac{1}{2}.$$

History $(0, 2)$ is symmetric. The effective prize spread is

$$v_A(0,2) = v_B(0,2) = v(1) - v(0) \geq 0.$$

Their effort supply equals

$$E[\widetilde{x_A}(0,2)] = E[\widetilde{x_B}(0,2)] = \frac{v(1) - v(0)}{2}.$$

The winning probabilities are

$$p_A(0,2) = p_B(0,2) = \frac{1}{2}.$$

When $(n_A, n_B) = (1, 1)$, the common effective prize spread is

$$v_A(1,1) = v_B(1,1) = v(2) - v(1) \geq 0.$$

Therefore, their expected effort supply equals

$$E[\widetilde{x_A}(1,1)] = E[\widetilde{x_B}(1,1)] = \frac{v(2) - v(1)}{2}.$$

The probabilities are

$$p_A(1,1) = p_B(1,1) = \frac{1}{2}.$$

We now consider the second battle. When $(n_A, n_B) = (1, 0)$, the effective prize spreads are as follows. For player 1,

$$\begin{aligned}\widetilde{v_A}(1,0) &= [p_A(2,0)v(3) + p_B(2,0)v(2) - E\widetilde{x_A}(2,0)] - [p_A(1,1)v(2) + p_B(1,1)v(1) - E\widetilde{x_A}(1,1)] \\ &= v(2) - v(1).\end{aligned}$$

Similarly,

$$\widetilde{v_B}(1,0) = v(1) - v(0).$$

We have $\widetilde{v_A}(1,0) \geq \widetilde{v_B}(1,0)$ if and only if $v(2) - v(1) \geq v(1) - v(0)$, which is the case considered in this lemma, i.e., prize profiles in $\mathcal{V}_4$.

The winning probabilities are

$$p_A(1,0) = 1 - \frac{1}{2}q,\ p_B(1,0) = \frac{q}{2},$$

where $q = \frac{v(1)-v(0)}{v(2)-v(1)}$.

Players' expected effort is

$$E[\widetilde{x_A}(1,0)] = \frac{1}{2}(v(1) - v(0)),\ E[\widetilde{x_B}(1,0)] = \frac{1}{2}\frac{(v(1) - v(0))^2}{(v(2) - v(1))}.$$

History $(0, 1)$ is a dual case of history $(1, 0)$. We have effort supply

$$E[\widetilde{x_A}(0,1)] = E[\widetilde{x_B}(1,0)] = \frac{1}{2}\frac{(v(1) - v(0))^2}{(v(2) - v(1))},\ E[\widetilde{x_B}(0,1)] = E[\widetilde{x_A}(1,0)] = \frac{1}{2}(v(1) - v(0)).$$

The winning probabilities are

$$p_A(0,1) = \frac{1}{2}q,\ \ p_B(0,1) = 1 - \frac{1}{2}q.$$

Now we come to the first battle. The common effective prize spread is

$$\begin{aligned}
&v_A(0,0) = v_B(0,0)\\
&= \{p_A(1,0)[p_A(2,0)v(3) + p_B(2,0)v(2) - E[\widetilde{x_A}(2,0)]]\\
&\qquad\qquad + p_B(1,0)[p_A(1,1)v(2) + p_B(1,1)v(1) - E[\widetilde{x_A}(1,1)]] - E[\widetilde{x_A}(1,0])\}\\
&\quad - \{p_A(0,1)[p_A(1,1)v(2) + p_B(1,1)v(1) - E[\widetilde{x_A}(1,1)]\\
&\qquad\qquad + p_B(0,1)[p_A(0,2)v(1) + p_B(0,2)v(0) - E[\widetilde{x_A}(0,2)] - x_A(0,1)\}\\
&= (1 - \frac{1}{2}q)(v(2) - v(0)) + (-\frac{1}{2} + \frac{q}{2})(v(1) - v(0)).
\end{aligned}$$

Therefore, their expected effort supply equals

$$E[\widetilde{x_A}(0,0)] = E[\widetilde{x_B}(0,0)] = \frac{v_A(0,0)}{2}.$$

and the winning probabilities are

$$p_A(0,0) = p_B(0,0) = \frac{1}{2}.$$

Total effort, therefore, is as follows:

$$\begin{aligned}
TE_4 &= 2E[\widetilde{x_A}(0,0)] + E[\widetilde{x_A}(1,0)] + E[\widetilde{x_B}(1,0)]\\
&\qquad + p_A(1,0)(E[\widetilde{x_A}(2,0)] + E[\widetilde{x_A}(2,0)]) + p_B(1,0)(E[\widetilde{x_A}(1,1)] + E[\widetilde{x_A}(1,1)])\\
&= v_A(0,0) + \frac{1}{2}(1+q)(v(1) - v(0)) + (1 - \frac{q}{2})(v(1) - v(0)) + \frac{q}{2}(v(2) - v(1))\\
&= 2(v(1) - v(0)) + (v(2) - v(1))\\
&= v(2) + v(1) - 2v(0)\\
&= 1 - 2v(0).
\end{aligned}$$

Maximizing the total effort TE_4 among prize structures in $\mathcal{V}_4$ yields the optimal allocations $v(0) = 0$, $v(2) \in [\frac{2}{3}, 1]$, $v(1) = 1 - v(2)$, and $v(3) = 1$. As a result, $TE_4^* = 1$, i.e., the rent is fully dissipated. ■

LEMMA 7 *When* $r > 2$, *the expected aggregate effort is* $TE_5 = 3(v(2) - v(1))$ *for all prize structures in* $\mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\}$. *The maximum aggregate effort in* $\mathcal{V}_5$ *equals* 1, *which is achieved by the prize structure* $(0, \frac{1}{3}, \frac{2}{3}, 1)$.

Proof. We use backward induction to solve the game. Note Lemma 1(iii) applies to all three battles. The third-battle results for history $(2,0)$, $(1,1)$ and $(0,2)$ remain same as in the proof of Lemma 6. Next, we look at the second battle. The expressions for the prize spreads remain same as in the proof of Lemma 6.

History $(1,0)$: $\widetilde{v_A}(1,0) \leq \widetilde{v_B}(1,0)$ if and only if $v(2) - v(1) \leq v(1) - v(0)$, and we are considering V_5.

Since their levels of effort follow

$$\widetilde{x_A}(1,0)=\mu^*,\ \widetilde{x_B}(1,0)=\begin{cases}\mu^*, & \text{with probability } q=\frac{v(2)-v(1)}{v(1)-v(0)},\\ 0, & \text{with probability } 1-q,\end{cases}$$

over $[0, v(2)-v(1)]$.
Their expected effort equals

$$E[\widetilde{x_A}(1,0)]=E[\mu^*]=\frac{1}{2}(v(2)-v(1)),$$

$$E[\widetilde{x_B}(1,0)]=qE[\mu^*]=\frac{1}{2}\frac{(v(2)-v(1))^2}{(v(1)-v(0))}.$$

The winning probabilities are

$$p_A(1,0)=\frac{1}{2}q,\ p_B(1,0)=1-\frac{q}{2},$$

where $q=\frac{v(2)-v(1)}{v(1)-v(0)}$.
History $(0,1)$ is the dual case of history $(1,0)$.
Now we come to the first battle. We first pin down the common effective prize spread as follows:

$$v_A(0,0)=v_B(0,0)=\frac{1}{2}q(v(2)-v(0))+(\frac{1}{2}-\frac{q}{2})(v(2)-v(1)).$$

Thus their expected effort supply equals

$$E[\widetilde{x_A}(0,0)]=\ E[\widetilde{x_B}(1,0)]=\frac{v_A(0,0)}{2}.$$

The winning probabilities are

$$p_A(0,0)=p_B(0,0)=\frac{1}{2}.$$

Total effort thus is as follow:

$$\begin{aligned}TE_5&=2E[\widetilde{x_A}(0,0)]+E[\widetilde{x_A}(1,0)]+E[\widetilde{x_B}(1,0)]\\&\quad+p_A(1,0)(E[\widetilde{x_A}(2,0)]+E[\widetilde{x_A}(2,0)])+p_B(1,0)(E[\widetilde{x_A}(1,1)]+E[\widetilde{x_A}(1,1)])\\&=v_A(0,0)+(\frac{1}{2}+q)(v(2)-v(1))-\frac{q}{2}(v(1)-v(0))+(1-\frac{q}{2})(v(2)-v(1))\\&=3(v(2)-v(1)).\end{aligned}$$

Therefore, the designer's problem is

$$
\begin{aligned}
\max \quad & TE_5 \\
s.t. \quad & (v(1) - v(0)) \geq (v(2) - v(1)), \\
& 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1, \\
& v(2) + v(0) \leq 1,
\end{aligned}
$$

which yields the optimal structure $v(0) = 0$, $v(1) = \frac{1}{3}$, $v(2) = \frac{2}{3}$ and $v(3) = 1$, and the optimal total expected effort $TE_5^* = 1$. ■

Note that $\mathcal{V} = \mathcal{V}_4 \cup \mathcal{V}_5$ when $r > 2$. Combining both lemmas immediately yields the following theorem, which fully characterizes the optimal prize profile when $r > 2$.

Appendix C: Proof of Theorem 4

Proof. We calculate the expected aggregate effort levels in lemmas 8 and 9. ■

LEMMA 8 *When $r \leq 2$, the expected total effort is $TE_T^1 = \frac{3r}{4}(v(2) - v(0))$ in the team contest.*

Proof. Note that Lemma 1(i) applies to all three battles when $r \leq 2$. Before a battle is fought, the history of past battles, or the state of the contest, is observed by players involved. The history of the contest is denoted (n_A, n_B) where n_A is the number of wins secured by team $i = A, B$. We solve the game by backward induction. We first look at the third battle.

History $(2, 0)$: the effective prize spreads are

$$v_A(2,0) = v(3) - v(2) \geq 0, v_B(2,0) = v(1) - v(0) \geq 0.$$

The budget constraint $v(3) + v(0) = v(2) + v(1)$ implies $v(3) - v(2) = v(1) - v(0)$, so $v_A(2,0) = v_B(2,0)$

By Lemma 1(i), we have effort supply

$$x_A(2,0) = \frac{rv_A^{r+1}(2,0)v_B^r(2,0)}{(v_A^r(2,0) + v_B^r(2,0))^2} = \frac{rv_A(2,0)}{4},$$

$$x_B(2,0) = \frac{rv_B^{r+1}(2,0)v_A^r(2,0)}{(v_A^r(2,0) + v_B^r(2,0))^2} = \frac{rv_B(2,0)}{4}.$$

The winning probabilities are

$$p_A(2,0) = \frac{x_A^r(2,0)}{x_A^r(2,0) + x_B^r(2,0)} = \frac{1}{2}, \ p_B(2,0) = \frac{x_B^r(2,0)}{x_A^r(2,0) + x_B^r(2,0)} = \frac{1}{2}$$

History $(0, 2)$ is similar. We now look at history $(1, 1)$. For history $(1, 1)$, the common prize spread is

$$v_A(1,1) = v_B(1,1) = v(2) - v(1).$$

Thus effort supply is

$$x_A(1,1) = x_B(1,1) = \frac{r}{4}(v(2) - v(1)).$$

The winning probabilities are

$$p_A(1,1) = p_B(1,1) = \frac{1}{2}.$$

We now look at the second battle. The history can be $(1, 0)$ or $(0, 1)$.

History $(1, 0)$: we first pin down the effective prize spreads:

$$\begin{aligned} v_A(1,0) &= [p_A(2,0)v(3) + p_B(2,0)v(2)] - [p_A(1,1)v(2) + p_B(1,1)v(1)] \\ &= \frac{1}{2}(v(3) - v(1)) = \frac{1}{2}(v(2) - v(0)), \end{aligned}$$

$$\begin{aligned} v_B(1,0) &= [p_B(1,1)v(2) + p_A(1,1)v(1)] - [p_B(2,0)v(1) + p_A(2,0)v(0)] \\ &= \frac{1}{2}(v(2) - v(0)). \end{aligned}$$

Thus effort supply is

$$x_A(1,0) = x_B(1,0) = \frac{r}{4}v_A(1,0) = \frac{r}{8}(v(2) - v(0)).$$

The winning probabilities are

$$p_A(1,0) = p_B(1,0) = \frac{1}{2}.$$

History $(0,1)$: This is dual case of history $(1,0)$.

Now we come to the first battle. We pin down the common effective prize spread:

$$\begin{aligned} & v_A(0,0) = v_B(0,0) \\ & = \{p_A(1,0)[p_A(2,0)v(3) + p_B(2,0)v(2)] + p_B(1,0)[p_A(1,1)v(2) + p_B(1,1)v(1)]\} \\ & \quad - \{p_A(0,1)[p_A(1,1)v(2) + p_B(1,1)v(1)] + p_B(0,1)[p_A(0,2)v(1) + p_B(0,2)v(0)]\} \\ & = \frac{1}{2}(v(2) - v(0)). \end{aligned}$$

Thus effort supply is

$$x_A(0,0) = x_B(0,0) = \frac{r}{4}v_A(0,0) = \frac{r}{8}(v(2) - v(0)).$$

The' winning probabilities are

$$p_A(0,0) = p_B(0,0) = \frac{1}{2}.$$

Thus, total effort can be calculated as follow:

$$\begin{aligned} TE_T^1 &= 2x_A(0,0) + [x_A(1,0) + x_B(1,0)] \\ & \quad + p_A(1,0)[x_A(2,0) + x_B(2,0)] + p_B(1,0)[x_A(1,1) + x_B(1,1)] \\ &= 2x_A(0,0) + 2x_A(1,0) + x_A(2,0) + x_A(1,1) \\ &= \frac{3r}{4}(v(2) - v(0)). \end{aligned}$$

■

LEMMA 9 *When $r > 2$, the expected total effort is $TE_T^2 = \frac{3}{2}(v(2) - v(0))$ in the team contest.*

Proof. Note Lemma 1(*iii*) applies to each battle. Moreover, Fu, Lu and Pan (2015) reveals that the prize spread is common for the two players in each battle. We solve the game by backward induction. We first look at the third battle.

History $(2,0)$: For history $(2,0)$, we first describe the two players' effective prize spreads:

$$v_A(2,0) = v(3) - v(2) \geq 0, v_B(2,0) = v(1) - v(0) \geq 0.$$

We have $v_A(2,0) = v_B(2,0)$ follows from the budget constraints (1).

Thus player's expected effort supply is given by

$$E[\widetilde{x_A}(2,0)] = E[\widetilde{x_B}(2,0)] = \frac{v(1) - v(0)}{2}.$$

The winning probabilities are

$$p_A(2,0) = p_B(2,0) = \frac{1}{2}.$$

Analogously, we analyze the history where $(0,2)$ and $(1,1)$.

Respectively, their effort supply follows

$$E[\widetilde{x_A}(0,2)] = E[\widetilde{x_B}(0,2)] = \frac{v(1) - v(0)}{2},$$

$$E[\widetilde{x_A}(1,1)] = E[\widetilde{x_B}(1,1)] = \frac{v(2) - v(1)}{2}.$$

History $(1,0)$: we first pin down the common effective prize spread:

$$v_B(1,0) = v_A(1,0) = \frac{1}{2}(v(2) - v(0)).$$

Their expected effort supply equals

$$E[\widetilde{x_A}(1,0)] = E[\widetilde{x_B}(1,0)] = \frac{v(2) - v(0)}{2}.$$

The winning probabilities are

$$p_A(1,0) = p_B(1,0) = \frac{1}{2}.$$

History $(0,1)$ is dual case of history $(1,0)$.

After we solve the first battle where history is $(0,0)$, their expected effort supply equals

$$E[\widetilde{x_A}(0,0)] = E[\widetilde{x_B}(0,0)] = \frac{v(2) - v(0)}{2}.$$

By symmetry, aggregate effort thus is as follow:

$$\begin{aligned} TE_T^2 &= 2E[\widetilde{x_A}(0,0)] + 2E[\widetilde{x_A}(1,0)] + E[\widetilde{x_A}(2,0)] + E[\widetilde{x_A}(1,1)] \\ &= \frac{3}{2}(v(2) - v(0)). \end{aligned}$$

■

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