Optimal Persuasion in First-price Auctions with Stochastic Entry*

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Abstract

In this paper, we adopt a Bayesian persuasion approach to study the seller's optimal policy for disclosing the actual number of bidders in first-price auctions with stochastic entry. We assume that the bidders' preference exhibits constant absolute risk aversion or risk loving. After entry, a participating bidder forms own posterior belief upon the public information released by the seller and his own entry status. We find that when bidders are risk averse (resp. risk loving), the revenue-maximizing policy is to fully conceal (resp. fully disclose) the actual number of participating bidders. When bidders are risk neutral, the induced expected revenue is invariant to the disclosure policy. We further show that the ex ante utility of each potential bidder does not depend on the disclosure policy. The seller-optimal policy is thus Pareto dominant. Our analytical procedure applies to alternative first-price and all-pay auction environments in which bidders are risk neutral and their disutility of payment is nonlinear, and similar results are shown to hold for the alternative auction environments.

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1 Introduction

It has long been recognized in the literature that in first-price auctions with a stochastic number of bidders, the information disclosed by the seller over the number of bidders directly affects entrants'

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bidding behaviors. The seller can thus use an appropriate disclosure policy as an effective instrument to control information release for her own interests. In particular, McAfee and McMillan (1987) and Matthews (1987) compare full-disclosure and full-concealment policies, and find that full concealment generates higher expected revenue when bidders are risk averse.

Clearly, there are many feasible disclosure policies other than full disclosure and full concealment, which have been studied by McAfee and McMillan (1987) and Matthews (1987). A question naturally arises: If we allow a more general class of policies, then what is the seller's optimal choice and how a policy affects the bidders' expected utility? In the context of online auctions, an auctioneer can choose whether or not to display the actual number of participating bidders. In fact, the auctioneer has more choices. For example, she could adopt a cutoff policy of disclosing the number of bidders if and only if it is above a certain threshold. In this paper, we examine this seller-optimal disclosure policy by comparing across all the feasible policies accommodated in a Bayesian persuasion framework. In particular, we will show how the optimal disclosure policy would depend on bidders' risk attitude. We will also study how the bidders' ex ante expected utility is affected by the disclosure policies.

We consider a first-price auction of a single object in the setting of McAfee and McMillan (1987). All bidders are symmetric ex ante, and their private values are their private information. We assume that bidders' preference exhibits constant absolute risk aversion or risk loving. All the players, including the seller and the bidders, have a common prior belief that each subgroup of bidders will be selected by nature as entrants with a group-specific probability. We assume that any two subgroups with the same size are selected as entrants with the same probability. After bidders' participation, the seller observes the number of entrants, however, each bidder only observes his own entry status unless the seller discloses more information. To best utilize her future superior information, the seller designs her optimal disclosure policy for revealing information about the number of actual bidders, and announces the policy publicly before nature makes the move. Following the Bayesian persuasion approach, we model the seller's disclosure policy as a signal-generating mechanism contingent on the realized number of actual entrants. Applying the Bayesian persuasion approach requires the seller's commitment power for her choice of disclosure policy. The sellers often have such commitment power in many situations. As the auction rules are typically announced in advance, a seller would follow the committed rule to maintain her credibility, even when ex post deviations are profitable. Moreover, in many online auctions, the disclosure policy is usually pre-programmed by the hosting web-pages, and deviation is usually infeasible.

To determine the seller's optimal disclosure policy, one must understand how a public signal affects entrants' bidding strategies. Since the entrant-generating process selects any two groups of bidders of same size with equal chance, any public signal realization will lead to a common posterior belief about the state of the contest (i.e., the number of entrants) among all entrants. Consequently, we focus on entrants' symmetric bidding strategy. In our problem, an entrant observes his own entry status in addition to the publicly released signal. Therefore, an entrant would update his own belief by combining these two sources of information. For a given disclosure policy, we then decompose an entrant's belief-updating process into two steps. In the first step, we derive the belief of a bidder who observes the signal realization but not his own entry status. Following Kamenica and Gentzkow (2011), all feasible distributions of first-step beliefs can be completely characterized. In the second step, we further express an entrant's posterior belief that bases on both the signal realization and his own entry status. This step delivers all the feasible entrants' common posteriors.

To solve for the seller's optimal disclosure policy, it is more convenient to work on the first-step belief. Therefore, by the relationship between the first- and second-step beliefs, we write the expected revenue under a particular signal realization as a function of the first-step belief. More importantly, we establish that the seller's ex ante expected revenue can still be computed by aggregating over the distribution of the first-step beliefs. The seller's optimization problem is thus to maximize the expected revenue by searching through all feasible distributions of first-step beliefs, which has been characterized by Kamenica and Gentzkow (2011).

To determine the optimal policy, we conduct the concavification analysis as in Kamenica and Gentzkow (2011) and Aumann and Maschler (1995). For this purpose, we examine the Hessian matrix of the expected revenue with respect to the first-step belief. The entrants' equilibrium bidding strategy and the seller's expected revenue under a common belief can be characterized explicitly, which facilitates the calculation of the concerned Hessian matrix. We therefore are able to determine the definiteness of the concerned Hessian matrix, through which we identify the optimal policy. When bidders are risk averse (resp. risk loving), the Hessian matrix is negative (resp. positive) semi-definite. Therefore, the seller's expected revenue is weakly concave (resp. convex) in terms of entrants' common posterior, which means that full concealment (resp. full disclosure) is the optimal policy. When bidders are risk neutral, the resulting Hessian matrix is a zero matrix, and therefore any disclosure policy yields the same expected revenue.

In addition, we show that the ex ante utility of each potential bidder is the same across all disclosure policies, regardless of the bidders' risk attitude. Therefore, the revenue-maximizing policy Pareto-dominates other policies, and the identified seller-optimal policy also maximizes the ex ante expected total surplus of all players.

We further study an alternative first price auction environment, in which bidders are risk neutral while their disutility of payment is nonlinear. We find our procedure still applies. When bidders' disutility function is convex (resp. concave), the seller's optimal policy is to fully conceal (resp. disclose). The same results are shown to hold for an all-pay auction with risk neutral bidders and nonlinear disutility of effort.[1]

Our paper is most closely related to McAfee and McMillan (1987) and Matthews (1987). They compare two polar policies (i.e., full disclosure and no disclosure) in the setting of first-price sealed-bid auctions with risk-averse bidders. They find that the seller prefers to conceal information about the number of entrants. Our results are consistent with theirs, as we find that full concealment is indeed the optimal policy among a whole range of feasible disclosure policies when bidders are constant absolute risk averse. Differentiating from their studies, we adopt a Bayesian persuasion perspective, which makes it feasible to compare full disclosure with any partial disclosure policy, and establish the optimality of the two polar policies among a complete family of disclosure policies described by signal-generating mechanisms. Moreover, our study further covers the cases with constant absolute risk loving and risk-neutral bidders.

Stochastic entry has also been studied recently in the contest literature. Lim and Matro (2009) and Fu, Jiao, and Lu (2011) adopt a Tullock contest setting with stochastic entry and compare the full-revealing and full-concealing policies from the perspective of the contest organizer. They find that the curvature of the characteristic function of the contest success function plays a key role in determining the preferred policy. Chen, Jiang and Knyazev (2015) compare the full-revealing and full-concealing policies in an all-pay auction setting with stochastic entry. Feng and Lu (2016) further adopt a Bayesian persuasion approach and study the optimal disclosure policy in a complete-information imperfectly discriminatory contest with stochastic entry. They demonstrate that if the characteristic function of the contest technology is strictly concave (resp. convex), the contest organizer's optimal policy is to fully reveal (resp. conceal) the number of actual contestants.

Our study is facilitated by the recent development in the Bayesian persuasion approach. Brocas and Carrillo (2007), Rayo and Segal (2010) and Kamenica and Gentzkow (2011) study how a sender strategically releases information to a receiver. Kamenica and Gentzkow (2011) formulate their problem in a convenient form by writing the sender's ex ante payoff as a function of the distribution of posterior beliefs, and develop a concavification technique to identify the optimal persuasion. Wang (2013) studies persuasions with multiple receivers in a voting game. Chan, Li, and Wang (2015) further study a more sophisticated Bayesian persuasion game between a sender and a

[1] Please refer to Appendix B for detailed analysis of all-pay auction.

set of voters. Li and Norman (2015) consider both sequential and simultaneous moves in a class of multi-sender persuasion games. Taneva (2016) considers a model of symmetric information where a designer chooses and announces the information structure about a payoff relevant state. She fully characterizes the designer's optimal choice of information structure in a two-state, two-agent, and two-action environment. Zhang and Zhou (2016) apply the Bayesian persuasion approach in a Tullock contest to study how to optimally influence an uninformed contestant's belief about his opponent's valuation. Ely, Frankel, and Kamenica (2015) further study optimal information disclosure to maximize expected suspense and surprise in dynamic environments. Our paper exemplifies another successful application of the Bayesian persuasion approach in a well-known environment of first-price auctions with multiple receivers and an uncertain number of entrants.

The rest of the paper is organized as follows. In Section 2, we set up the incomplete-information first-price auction model with stochastic entry and risk averse/loving bidders. In Section 3, we present the analysis of optimal disclosure policy in the environment of Section 2. In Section 4, we study an alternative first price auction environment, in which bidders are risk neutral while their disutility of payment is nonlinear. Section 5 provides a concluding remark. Appendix A collects the proofs for the first-price auction environments, and Appendix B studies the optimal disclosure policy in an all-pay auction environment with risk neutral bidders and nonlinear disutility of effort.

2 The model setup

We consider a first-price auction with a single item, M potential bidders, and one seller. The bidders' private values are independently drawn from a common cumulative distribution $F(\cdot)$ with density function $f(\cdot)$, where f is continuous and strictly positive on its support $[0, \overline{x}]$. The values are bidders' private information, while the value distribution is public information.

We assume that the bidders have the following exponential utility function.

Assumption 1 $u(w) = \frac{1-e^{-\lambda w}}{\lambda}, \lambda \in (-\infty, +\infty)$.

When $\lambda > 0$, $u(\cdot)$ exhibits constant absolute risk aversion; when $\lambda < 0$, $u(\cdot)$ exhibits constant absolute risk loving. When $\lambda \to 0$, $u(w)$ converges to $u_0(w) = w$, which exhibits risk neutrality.

It is common knowledge among the potential bidders and the seller that a subset $A \in 2^M$ will be selected by nature with probability $\mu_0(A)$ to participate in the first-price auction. We have $\sum_{A \in 2^M} \mu_0(A) = 1$. Therefore, $\{\mu_0(A), \forall A \in 2^M\}$ is common prior belief of all the players. We assume the following assumption:

Assumption 2 *(Symmetry in entry)* $\forall A, A' \in 2^M$, *if* $|A| = |A'|$, *we have* $\mu_0(A) = \mu_0(A')$.

Assumption 2 says that any two groups with the same size have the same participating probability, which means that what will affect the participating probability of a group is the size of the group, instead of the bidders' identities in that group. In particular, it accommodates the case in which each contestant has a fixed and independent participating probability. Under Assumption 2, we can then focus on the bidders' belief about the number of entrants. Put it differently, when Assumption 2 violates, what matters is the bidders' belief about the subsets of entrants rather than belief about the numbers of entrants, as a public disclosure policy may induce different posteriors.

In our model, we allow the seller to learn the number of entrants after nature selects the participating group, however, an entrant only observes his own participation status but not other entrants' participation, unless the seller discloses additional information. All selected bidders bid simultaneously based on their posterior beliefs about the number of actual bidders. The highest bidder wins and pays his own bid. The losing bidders do not pay.

The seller aims to maximize her expected revenue by choosing how to disclose information about the actual number of bidders. The seller's disclosure policy affects entrants' bidding behavior by influencing their posterior beliefs. Let $S = \Omega = \{0, 1, 2, ..., M\}$.[2] A disclosure policy of the seller is described by $\{\pi(\cdot|N)\}_{N \in \Omega}$ over the signal space S, for each possible cardinality $N (\in \Omega)$ of a participating group. In the event that N bidders participate, a publicly observed signal $s \in S$ is generated following the distribution $\pi(\cdot|N)$. In particular, policy $\pi(s|N) = \mathbf{1}_{\{(s,N)|s=N\}}(s, N)$ leads to full disclosure, and policy $\pi(s|N) = \frac{1}{M+1}$, $\forall s, N$, corresponds to full concealment.

We assume that the seller has precommitment power. In other words, the seller will stick to any disclosure policy that has been preannounced before the entrants' participation. This commitment power is required by the Bayesian persuasion approach we adopt in this paper.

3 The analysis of optimal disclosure policy

In this section, we present the analysis of optimal disclosure policies for the Section 2 environment. In Section 3.1, we first identify entrants' common posterior belief conditional on a public signal realization generated by a disclosure policy. In Section 3.2, assuming that entrants hold a common posterior, we analyze their equilibrium behavior and calculate the expected revenue. In Section

[2] As shown in Kamenica and Gentzkow (2011) and Zhang and Zhou (2015), it is without loss of generality to assume that the size of the signal space is less than the minimum of the size of action space and the state space. In our model, the action space is continuous and the state space is $\Omega = \{0, 1, 2, ..., M\}$.

3.3, we show that the observability of the actual number of entrants by the seller after bidders' entries makes no difference to the formulation of her expected revenue. Combining the results in Sections 3.1 to 3.3, we formulate the seller' optimization problem on her choice of disclosure policy. In Section 3.4, we solve for the optimal disclosure policies from both the perspectives of seller and buyers.

3.1 Belief updating

A representative entrant i observes the signal realization s and his own entry status, and forms his own posterior belief $\mu(\cdot|s,i)$ over all possible numbers of entrants. Given that bidder i is an entrant, what is his belief about the actual entrants without any additional information? $\forall A_i \in 2^M$, where $i \in A_i$, given i's entry, his conditional belief is

$$\mu_0(A_i|i) = \frac{\mu_0(A_i)}{\sum_{\forall A_i \in 2^M} \mu_0(A_i)}, \forall A_i,$$

and $\mu_0(A|i) = 0$, $\forall A \in 2^M$ such that $i \notin A$.

Conditional on the additional information s, an entrant i's posterior belief $\mu(A|s,i)$ about $A \in 2^M$ is as follows: $\mu(A|s,i) = 0, \forall i \notin A$; and

$$\mu(A_i|s,i) = \frac{\pi(s||A_i|)\mu_0(A_i)}{\sum_{\forall A_i \in 2^M} \pi(s||A_i|)\mu_0(A_i)}, \forall A_i.$$

We can then further identify his belief $\mu(N|s,i)$ about $N \in \{0,1,2,...,M\}$ after observing s and his own entry as follows: $\mu(0|s,i) = 0$ and

$$\mu(N|s,i) = \sum_{\forall A_i \in 2^M, |A_i|=N} \mu(A_i|s,i).$$

Under Assumption 2, no entrant can deduce more information than other entrants based on his own entry status and the public signal s. Therefore, this simplification assumption enables us to focus our analysis on the belief about the number of entrants. In the following lemma, we establish the relationship between an entrant's posterior belief and the signal realization. Since $\mu(N|s,i)$ does not depend on the identity of an entrant, we can denote the common belief by $\mu(N|s,e)$, where e is for entrants. A common belief across all entrants not only simplifies the belief-updating process, but also greatly facilitates the equilibrium analysis.

We let $\mu_0(N) = \sum_{\forall A \in 2^M, |A|=N} \mu_0(A)$, and let $\mu_s(N) = \mu(N|s) = \frac{\pi(s|N)\mu_0(N)}{\sum_{N'=0}^{M} \pi(s|N')\mu_0(N')}, \forall N \in \Omega$ stand for an entrant's updated belief based only on the signal realization s, without knowing his own entry status.

Lemma 1 *Under Assumption 2, every entrant observing a signal s and his own entry holds a common posterior belief*

$$\mu(N|s,e) = \mu_e(N|\mu_s) = \frac{N\mu_s(N)}{\sum_{N'=1}^{M} N'\mu_s(N')}, \forall N \in \{0,1,...,M\}. \tag{1}$$

The proof for Lemma 1 is in Appendix A. Lemma 1 establishes the connection between the common posterior belief $\{\mu(\cdot|s,e)\}$ of entrants and the belief $\{\mu_s(\cdot)\}$, which is updated only upon signal s. It shows that $\{\mu(\cdot|s,e)\}$ can be fully pinned down by $\{\mu_s(\cdot)\}$. This connection will be used to show, in Proposition 1, that expected revenue can be written as a function of belief $\{\mu_s(\cdot)\}$, which greatly facilitates the search for the optimal disclosure policy.

Since the signal s affects $\mu(N|s,e)$ solely through μ_s, we use $\mu_e(\cdot|\mu_s)$ hereafter to denote belief $\mu(\cdot|s,e)$ to simplify notation.

3.2 Symmetric equilibrium under a given common belief

Given a public signal s generated under a given disclosure policy, under Assumption 2, entrants form a common posterior belief about the number of participants, as established in Section 3.1. We next identify the symmetric equilibrium bidding function of the entrants for any given common posterior belief $\{\mu_e(\cdot|\mu_s), N = 1,2,...,M\}$.

3.2.1 Risk-averse and risk-loving bidders: $\lambda \neq 0$

We focus on characterizing the unique symmetric pure strategy Bayesian Nash equilibrium. Given participating bidders' common posterior belief $\mu_e(\cdot|\mu_s)$, let $B(x, \mu_e(\cdot|\mu_s))$ be the corresponding equilibrium bidding function, which is strictly increasing.

Lemma 2 *For $\lambda \neq 0$, if participating bidders hold the same posterior belief $\mu_e(N|\mu_s)$ and Assump-*

tions 1 and 2 hold, the entrants' unique symmetric equilibrium bidding strategy is

$$B(x, \mu_e(\cdot|\mu_s)) = \frac{1}{\lambda} \ln \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1)e^{\lambda t} F^{N-2}(t)f(t)dt] - \frac{1}{\lambda} \ln \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x).$$

Without loss of generality, we assume that $\mu_0 \in int(\Delta^M)$, where $\Delta^M = \{\mu \in R^M | \mu(N) \geq 0, \forall N \in \Omega = \{0, 1, ..., M\}$ and $\sum_{N=0}^{M} \mu(N) = 1\}$.[3]

Combining Lemmas 1 and 2, we have the following proposition, which shows that the equilibrium strategy can be alternatively written as a function of belief μ_s.

Proposition 1 *Under Assumptions 1 and 2, for $\lambda \neq 0$, given a disclosure policy π, conditional on a signal realization s, entrants' unique symmetric equilibrium bidding strategy is*

$$\sum_{N=1}^{M} N\mu_s(N)F^{N-1}(x)e^{\lambda B(x, \mu_e(\cdot|\mu_s))} = \sum_{N=1}^{M} N\mu_s(N)[\int_0^x (N-1)e^{\lambda t} F^{N-2}(t)f(t)dt]. \tag{2}$$

Recall that $\mu_s(N) = \mu(N|s) = \frac{\pi(s|N)\mu_0(N)}{\sum_{N'=0}^{M} \pi(s|N')\mu_0(N')}, \forall N \in \{0, 1, ..., M\}$ is a bidder's updated belief based only on the signal realization s (without knowing his own entry status). Alternatively, we can write the bidding function given by (2) explicitly:

$$B(x, \mu_e(\cdot|\mu_s)) = \frac{1}{\lambda} \ln \sum_{N=1}^{M} N\mu_s(N)[\int_0^x (N-1)e^{\lambda t} F^{N-2}(t)f(t)dt] - \frac{1}{\lambda} \ln \sum_{N=1}^{M} N\mu_s(N)F^{N-1}(x).$$

3.2.2 Risk-neutral bidders

We now turn to the case where all bidders are risk neutral. In this case, all bidders have the same utility function

$$u(w) = w.$$

Analogous to the equilibrium analysis for risk-averse bidders, we solve for the unique pure strategy equilibrium bidding function $B(x, \mu_e(\cdot|\mu_s))$ of a risk-neutral entrant with belief $\mu_e(\cdot|\mu_s)$

[3] When $\mu_0 \notin int(\Delta^M)$, we can simply reduce the dimension of M as long as μ_0 is not degenerate. However, we exclude degenerate cases.

and value $x \in [0, \overline{x}]$.

Lemma 3 *When bidders are risk neutral, if participating bidders hold the same posterior belief $\mu_e(\cdot|\mu_s)$ and Assumptions 1 and 2 hold, entrants' unique symmetric equilibrium bidding strategy is*

$$\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) B(x, \mu_e(\cdot|\mu_s)) = \sum_{N=1}^{M} \mu_e(N|\mu_s) \int_0^x (N-1) t F^{N-2}(t) f(t) dt.$$

Combining Lemmas 1 and 3, we have the following proposition, which gives the equilibrium strategy as a function of belief μ_s.

Proposition 2 *Under Assumptions 1 and 2, when bidders are risk neutral, given a disclosure policy π, conditional on a signal realization s, entrants' unique symmetric equilibrium bidding strategy is*

$$\sum_{N=1}^{M} N \mu_s(N) F^{N-1}(x) B(x, \mu(\cdot|s, e)) = \sum_{N=1}^{M} N \mu_s(N) \int_0^x (N-1) t F^{N-2}(t) f(t) dt. \tag{3}$$

Alternatively, we can write the bidding function given by (3) explicitly:

$$B(x, \mu_e(\cdot|\mu_s)) = \frac{\sum_{N=1}^{M} N \mu_s(N) \int_0^x (N-1) t F^{N-2}(t) f(t) dt}{\sum_{N=1}^{M} N \mu_s(N) F^{N-1}(x)}.$$

3.3 The seller's problem

In our setting, the seller observes the number of the entrants (i.e., the true state) after nature selects the set of participants. This is different from Kamenica and Gentzkow's (2011) setup, in which both the sender and the receiver do not observe the true state. Nevertheless, we next establish that Kamenica and Gentzkow's formula for the sender's ex ante expected payoff remains applicable in our context, in which the seller is the sender and the bidders are the receivers.

Let μ denote the belief of a bidder who only observes the public signal but not his entry status.

Conditional on N and μ, the sender's payoff is

$$\begin{aligned}
\widetilde{R}(\mu, N) \quad : \quad &= R(\mu_e(\cdot|\mu), N) \\
&= \int_0^{\overline{x}} B(x, \mu_e(\cdot|\mu)) dF^N(x) \\
&= N \int_0^{\overline{x}} F^{N-1}(x) f(x) B(x, \mu_e(\cdot|\mu)) dx.
\end{aligned}$$

Let

$$\begin{aligned}
R(\mu) &= E_\mu[\widetilde{R}(\mu, N)] \\
&= \sum_{N=1}^{M} \mu(N) \int_0^{\overline{x}} N F^{N-1}(x) f(x) B(x, \mu_e(\cdot|\mu)) dx. \qquad (4)
\end{aligned}$$

The following lemma gives the seller's ex ante expected revenue as a function of distribution τ for the belief μ.

Lemma 4 *Given a disclosure policy π that induces a distribution τ for the belief μ of a bidder who only observes the public signal but not his own entry status, the seller's ex ante expected revenue equals*

$$E_\tau R(\mu).$$

We thus can write the seller's problem as follows:

$$\max_\tau E_\tau R(\mu)$$

$$s.t. \sum_{\mu \in \text{supp}(\tau)} \tau(\mu)\mu = \mu_0,$$

where $\mu = (\mu(1), \mu(2), ..., \mu(M))$ is of M dimensions as $\sum_{N=1}^{M} \mu(N) = 1 - \mu(0)$, and τ denotes the distribution of posteriors μ. The above usual constraint on τ, as specified in Kamenica and Gentzkow (2011), applies.

We would like to emphasize that Lemma 4 is also applicable in the environments of Section 4 and Appendix B, in which the equilibrium bidding strategy $B(x, \mu_e(\cdot|\mu))$ in $R(\mu)$ takes different forms though.

3.4 The optimal disclosure policy

We will first solve for the seller's optimal disclosure policy in Theorem 1. We will further study how the bidders' ex ante utility depends on the disclosure policies in Theorem 2.

Theorem 1 *Under Assumptions 1 and 2, we have*
(i) the optimal disclosure policy is full concealment if bidders are risk averse, i.e., $\lambda > 0$*;*
(ii) the optimal disclosure policy is full disclosure if bidders are risk loving, i.e., $\lambda < 0$*; and*
(iii) the seller's ex ante expected revenue is invariant to the disclosure policy if bidders are risk neutral.

We prove Theorem 1 by showing that $R(\mu)$ is concave (resp. convex) when $\lambda > 0$ (resp. $\lambda < 0$). The proof for Theorem 1 is relegated to Appendix A.

The optimality of full concealment policy is explained by what McAfee and McMillan (1987) called the bid-dispersion effect: revealing information results in a higher variance and therefore lowers the revenue on average. Theorem 1 shows that this bid-dispersion effect works when bidders are risk averse (i.e., $\lambda > 0$), but not for risk-loving bidders (i.e., $\lambda < 0$), and what drives the bid-dispersion effect is the concavity of $R(\mu)$. In fact, whether a higher variance is profitable for the seller, all depends on the convexity/concavity of $R(\mu)$, which is further determined by the bidders' risk attitude.

When bidders are risk averse, the seller's ex ante revenue $R(\mu)$ increases less in response to a more pessimistic belief μ, i.e., when entrants believe that more likely there is a large number of entrants. To see the reason, recall that, in a first-price auction, the risk-averse bidders will bid more aggressively than risk-neutral bidders in order to secure a win. In addition, a bidder never bids more than his true valuation in first-price auctions. Consequently, risk-averse bidders tend to react less sensitively to a more pessimistic belief μ, compared to risk neutral bidders. Therefore, the seller's ex ante revenue $R(\mu)$ responses less sensitively to a more pessimistic belief μ, that is, $R(\mu)$ is concave in μ, when $\lambda > 0$. The intuitions behind Theorem 1(*ii*) and 1(*iii*) can be analogously illustrated.

In the next theorem, we study how a disclosure policy affects the bidders' expected utility. We show that a bidder's ex ante expected utility actually does not depend on the disclosure policy in our context, regardless of the risk attitude of the bidders. This result immediately means that the seller-optimal policy must Pareto dominate other policies, and it renders the highest expected total surplus for all players.

Theorem 2 *(i) The ex ante utility of each potential bidder is invariant to the disclosure policy. (ii) The seller-optimal disclosure policy is Pareto dominant.*

4 First-price auction with nonlinear disutility of payment

In this section, we study an alternative first-price auction environment, which only differs from the original setup of Section 2 in that bidders are risk-neutral but their disutility of payment is nonlinear. Specifically, the winner suffers a disutility of $g(b)$ if he pays his bid b, where function $g(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ is twice continuously differentiable, $g'(\cdot) > 0$ and $g(0) = 0$. The same value distribution for bidders and the same entrant-generating process as specified in Section 2 prevail. To be consistent with the analysis for the original setup and avoid introducing more notations, we use the same set of notations as in Sections 2 and 3.

Given a representative participant i's posterior belief about the number of the participants $\mu_e(\cdot|\mu_s)$ and his value $x \in [0, \overline{x}]$, he maximizes his expected utility by choosing his bid $b:$

$$\max_b U(b, x) := E_{\mu_e(\cdot|\mu_s)}[F^{N-1}(B^{-1}(b))(x - g(b))].$$

The following lemma characterizes the unique symmetric pure strategy Bayesian Nash equilibrium. The proof is relegated to the Appendix A.

Lemma 5 *If participating bidders hold the same posterior belief $\mu_e(\cdot|\mu_s)$ and Assumption 2 holds, entrants' symmetric equilibrium bidding strategy $B(x, \mu_e(\cdot|\mu_s)$ is given by*

$$\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) g(B(x, \mu_e(\cdot|\mu_s))) = \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1)F^{N-2}(t)f(t)tdt].$$

As in Section 3, we assume $\mu_0 \in int(\Delta^M)$, where $\Delta^M = \{\mu \in R^M | \mu(N) \geq 0, \forall N \in \Omega = \{0, 1, ..., M\}$ and $\sum_{N=0}^{M} \mu(N) = 1\}$.

Combining Lemmas 1 and 5, we have the following proposition, which shows that the equilibrium strategy can be alternatively written as a function of belief μ_s.

Proposition 3 *Under Assumption 2, given a disclosure policy π, conditional on a signal realization s, entrants' unique symmetric equilibrium bidding strategy is*

$$\sum_{N=1}^{M} N\mu_s(N)F^{N-1}(x)g(B(x,\mu_e(\cdot|\mu_s))) = \sum_{N=1}^{M} N\mu_s(N)[\int_0^x (N-1)F^{N-2}(t)f(t)tdt]. \quad (5)$$

Alternatively, we can write the bidding function given by (5) explicitly:

$$B(x,\mu_e(\cdot|\mu_s)) = g^{-1}(\frac{\sum_{N=1}^{M} N\mu_s(N)[\int_0^x (N-1)F^{N-2}(t)f(t)tdt]}{\sum_{N=1}^{M} N\mu_s(N)F^{N-1}(x)}).$$

The sender's optimization problem can be formulated in the same way as in Section 3.3 except that the equilibrium bidding strategy now takes a different form. We solve for the optimal disclosure policy and summarize the results in the following Theorem. The proof of Theorem 3 is relegated to Appendix A.

Theorem 3 *When Assumption 2 holds and bidders are risk neutral, we have*
(i) the optimal disclosure policy is full concealment if g is convex;
(ii) the optimal disclosure policy is full disclosure if g is concave;
(iii) any policy renders the same total expected revenue if g is linear; and
(iv) the ex ante utility of each potential bidder is invariant to the disclosure policy.

The same set of results holds for all-pay auctions with risk neutral bidders and nonlinear bidding costs. To save space, the detailed analysis is presented in Appendix B.

5 Concluding remarks

Our study shows that the Bayesian persuasion approach can be successfully applied to examine the optimal policy for disclosing the number of actual participants in first-price auctions with stochastic entry. We find that the bidders' risk attitude plays a crucial role in determining the optimal disclosure policy. If bidders are constant absolute risk averse (resp. loving), the seller's optimal policy is to fully reveal (resp. conceal) the actual number of participants. If bidders are risk neutral, any disclosure policy yields the seller the same expected revenue. In addition, we show that the ex ante expected utility of each potential bidder is the same across the disclosure policies. As a result, the seller-optimal policy Pareto-dominates other policies. Our analysis extends to alternative first-price and all-pay auction environments with risk neutral bidders and nonlinear disutility of payment or effort.[4] When bidders' disutility function is convex (resp. concave), the seller's optimal policy is full concealment (resp. disclosure).

In this paper, entry is exogenously stochastic and disclosure policy does not affect bidders' entry behavior. In some situations, bidders incur an entry cost to participate. Therefore, entry is endogenously stochastic, and different disclosure policies may induce different entry behavior. As a result, the optimal disclosure policy must balance bidders' participation and bid-eliciting from entrants. An intriguing issue is whether our findings can be generalized to the environment with endogenous entry—and if not, what would be the optimal policy to generate the highest expected revenue. We leave this to future research.

[4] Moldovanu and Sela (2001) study all-pay auctions with incomplete information, and allow linear and nonlinear effort cost functions. If stochastic entry is introduced into their model, a similar procedure that used in this paper can be applied to study the optimal disclosure policy for non-linear effort cost functions. We present the analysis and results in Appendix B.

6 Appendix A: proofs for first-price auction environments

Proof of Lemma 1

Proof. Recall that prior belief about number of entrants $\mu_0(N) = \sum_{\forall A \in 2^M, |A|=N} \mu_0(A)$, $\forall N \in \Omega$. By Assumption 2, we have $\mu_0(A) = \frac{\mu_0(N)}{C_M^N}$ for all A such that $|A| = N$. Clearly, $\mu(0|s,i) = 0$. We now identify $\mu(N|s,i), \forall N \in \{1, ..., M\}$ under Assumption 2.

$$\begin{aligned}
\mu(N|s,i) &= \sum_{\forall A_i \in 2^M, |A_i|=N} \mu(A_i|s,i) \\
&= \sum_{\forall A_i \in 2^M, |A_i|=N} \frac{\pi(s||A_i|)\mu_0(A_i)}{\sum_{\forall A_i \in 2^M} \pi(s||A_i|)\mu_0(A_i)} \\
&= \frac{\sum_{\forall A_i \in 2^M, |A_i|=N} \pi(s||A_i|)\frac{\mu_0(|A_i|)}{C_M^{|A_i|}}}{\sum_{\forall A_i \in 2^M} \pi(s||A_i|)\frac{\mu_0(|A_i|)}{C_M^{|A_i|}}} \\
&= \frac{\sum_{\forall A_i \in 2^M, |A_i|=N} \pi(s|N)\frac{\mu_0(N)}{C_M^{N}}}{\sum_{\forall A_i \in 2^M} \pi(s||A_i|)\frac{\mu_0(|A_i|)}{C_M^{|A_i|}}} \\
&= \frac{\sum_{\forall A_i \in 2^M, |A_i|=N} \pi(s|N)\frac{\mu_0(N)}{C_M^{N}}}{\sum_{N'=1}^{M} \sum_{\forall A_i \in 2^M, |A_i|=N'} \pi(s|N')\frac{\mu_0(N')}{C_M^{N'}}} \\
&= \frac{C_{M-1}^{N-1}\pi(s|N)\frac{\mu_0(N)}{C_M^{N}}}{\sum_{N'=1}^{M} C_{M-1}^{N'-1}\pi(s|N')\frac{\mu_0(N')}{C_M^{N'}}} \\
&= \frac{\pi(s|N)\mu_0(N)\frac{N}{M}}{\sum_{N'=1}^{M} \pi(s|N')\mu_0(N')\frac{N'}{M}} \\
&= \frac{\pi(s|N)\mu_0(N)N}{\sum_{N'=1}^{M} \pi(s|N')\mu_0(N')N'}.
\end{aligned}$$

Recall that $\mu_s(N) = \mu(N|s) = \frac{\pi(s|N)\mu_0(N)}{\sum_{N'=0}^{M} \pi(s|N')\mu_0(N')}$. We thus have

$$\begin{aligned}\mu(N|s,i) &= \frac{\pi(s|N)\mu_0(N)N}{\sum_{N'=1}^{M} \pi(s|N')\mu_0(N')N'} \\ &= \frac{N\mu_s(N)}{\sum_{N'=1}^{M} N'\mu_s(N')}.\end{aligned}$$

Note that $\mu(N|s,i)$ is independent of i and $\mu(0|s,i) = 0$. From now on, we write $\mu(N|s,e) := \mu(N|s,i) = \frac{N\mu_s(N)}{\sum_{N'=1}^{M} N'\mu_s(N')}, \forall N \geq 0$. ■

Proof of Lemma 2

Proof. Given participant i's posterior belief about the number of participants $\mu_e(\cdot|\mu_s)$ and his value $x \in [0, \overline{x}]$, he maximizes his expected utility by choosing his bid b :

$$\max_b U(b) := E_{\mu_e(\cdot|\mu_s)}[F^{N-1}(B^{-1}(b))\frac{1-e^{-\lambda(x-b)}}{\lambda}],$$

i.e.,

$$\max_b \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(B^{-1}(b))\frac{1-e^{-\lambda(x-b)}}{\lambda}].$$

At a Nash equilibrium:

$$\frac{dU(b)}{db}|_{b=B(x)} = 0, \tag{6}$$

which gives

$$\begin{aligned}&\sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x)e^{-\lambda(x-B(x))}B'(x) \\ =& \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(x)f(x)\frac{1-e^{-\lambda(x-B(x))}}{\lambda}.\end{aligned}$$

Let

$$EU(x) \quad : \quad = EU(B(x)) \tag{7}$$
$$= \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) \frac{1}{\lambda}(1 - e^{-\lambda(x-B(x))}).$$

Applying the envelope theorem, we have

$$\frac{d}{dx}(EU(x)) = \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-1}(x)e^{-\lambda(x-B(x))}$$
$$= \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x) - \lambda EU(x).$$

Solving this linear differential equation for $EU(x)$, we have

$$EU(x) = e^{-\lambda x}[K + \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(t)e^{\lambda t}dt],$$

where K is a constant, which can be identified from boundary condition $EU(x)|_{x=0} = 0$. We thus have $K = 0$. Therefore,

$$EU(x) = e^{-\lambda x}[\int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(t)e^{\lambda t}dt]. \tag{8}$$

From (7) and (8), we have

$$\sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x)\frac{1}{\lambda}(1 - e^{-\lambda(x-B(x)})$$
$$= \sum_{N=1}^{M} \mu_e(N|\mu_s)e^{-\lambda x} \int_0^x F^{N-1}(t)e^{\lambda t}dt$$
$$= \sum_{N=1}^{M} \mu_e(N|\mu_s)e^{-\lambda x}[\frac{1}{\lambda}F^{N-1}(t)e^{\lambda t}|_0^x - \frac{1}{\lambda}\int_0^x (N-1)e^{\lambda t}F^{N-2}(t)f(t)dt].$$

Thus, the bidding function $B(x, \mu_e(\cdot|\mu_s))$ of an entrant with belief $\mu_e(\cdot|\mu_s)$ and value x satisfies:

$$\begin{aligned} & \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) e^{-\lambda(x - B(x,\mu_e(\cdot|\mu_s)))} \\ = \quad & e^{-\lambda x} \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt], \end{aligned}$$

or equivalently

$$\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) e^{\lambda B(x,\mu_e(\cdot|\mu_s))} = \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt].$$

In particular, at $x = 0$, we know that $B(0, \mu_e(\cdot|\mu_s)) = 0$, which also satisfies the above equation because of L'Hôpital's rule.

■

Proof of Lemma 3

Proof. Given participant $i's$ posterior belief about the number of participants $\mu_e(\cdot|\mu_s)$ and value $x \in [0, \overline{x}]$, he maximizes his expected utility by choosing his bid b

$$\max_b U(b) := E_{\mu_e(\cdot|\mu_s)}[F^{N-1}(B^{-1}(b))(x - b)],$$

i.e.,

$$\max_b \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(B^{-1}(b))(x - b)].$$

At a Nash equilibrium:

$$\frac{dU(b)}{db}|_{b=B(x)} = 0.$$

We thus have

$$\sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1) F^{N-2}(x) f(x)(x - B(x)) = \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) B'(x). \tag{9}$$

Let

$$\begin{aligned} EU(x) \quad : \quad &= EU(B(x)) \\ &= \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x)(x - B(x)). \end{aligned} \tag{10}$$

Using (10), we thus have

$$\begin{aligned} \frac{d}{dx}(EU(x)) &= \sum_{N=1}^{M} \mu_e(N|\mu_s)[(N-1)F^{N-2}(x)f(x)(x - B(x)) \\ &\quad + F^{N-1}(x)(1 - B'(x))] \\ &= \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x). \end{aligned}$$

Solving this linear differential equation for $EU(x)$:

$$EU(x) = K + \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(t) dt,$$

where K is a constant that can be indentified by boundary condition $EU(x)|_{x=0} = 0$. Therefore,

$$EU(x) = \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(t) dt. \tag{11}$$

From (10) and (11),

$$\begin{aligned} & \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x)(x - B(x)) \\ = & \sum_{N=1}^{M} \mu_e(N|\mu_s) \int_0^x F^{N-1}(t) dt \\ = & \sum_{N=1}^{M} \mu_e(N|\mu_s)[tF^{N-1}(t)|_0^x - (N-1) \int_0^x tF^{N-2}(t) f(t) dt]. \end{aligned}$$

Thus, the bidding function $B(x, \mu_e(\cdot|\mu_s))$ of an entrant with belief $\mu_e(\cdot|\mu_s)$ and value x satisfies:

$$\begin{aligned} & \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) B(x, \mu_e(\cdot|\mu_s)) \\ = & \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1) t F^{N-2}(t) f(t) dt], \end{aligned}$$

or

$$\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) B(x, \mu_e(\cdot|\mu_s)) = \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1) t F^{N-2}(t) f(t) dt].$$

Alternatively,

$$B(x, \mu_e(\cdot|\mu_s)) = \frac{\sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1) t F^{N-2}(t) f(t) dt]}{\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x)}.$$

■

Proof of Lemma 4

Proof. In our model, the state is denoted by N, the number of actual participants. Recall that $\mu_s(N) = \frac{\pi(s|N)\mu_0(N)}{\sum_{N'} \pi(s|N')\mu_0(N')}$ is the posterior belief when a receiver observes the public signal realization s. Conditional on N and s, the seller's expected revenue is

$$\begin{aligned} \widetilde{R}(\mu_s, N) \quad : & = R(\mu_e(\cdot|\mu_s), N) \\ & = N \int_0^{\bar{x}} F^{N-1}(x) f(x) B(x, \mu_e(\cdot|\mu_s)) dx, \end{aligned}$$

where $B(x, \mu_e(\cdot|\mu_s))$ is the bidders' bidding strategy that satisfies (2) (or (3)).

Let τ denote the distribution of posteriors μ. Combining the above results, the seller maximizes the expected revenue by choosing τ:

$$\max_{\tau} E_\tau E_\mu \widetilde{R}(\mu_e(\cdot|\mu), N)$$

$$s.t. \sum_{\mu \in \text{supp}(\tau)} \tau(\mu)\mu = \mu_0.$$

The usual constraint, as specified in the above problem, applies as in Kamenica and Gentzkow (2011).

Let $R(\mu) = E_\mu[\widetilde{R}(\mu_e(\cdot|\mu), N)]$, i.e.,

$$\begin{aligned} R(\mu) &= E_\mu[\widetilde{R}(\mu, N)] \\ &= \sum_{N=1}^{M} \mu(N) \int_0^{\overline{x}} N F^{N-1}(x) f(x) B(x, \mu_e(\cdot|\mu)) dx. \end{aligned}$$

■

Proof of Theorem 1

Proof. We first prove parts (i) and (ii) for $\lambda \neq 0$. Recall that, from Proposition 1, we have

$$\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) e^{\lambda B(x, \mu_e(\cdot|\mu))} = \sum_{N=1}^{M} N\mu(N)[\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt],$$

where $\mu = (\mu(1), \mu(2), ..., \mu(M))$ is of M dimension as $\sum_{j=1}^{M} \mu(j) = 1 - \mu(0)$.

Taking the second order derivative wrt. $\mu(i)$ and $\mu(j)$ on both sides for $\forall i, j \in \{1, 2, ..., M\}$, we obtain

$$\frac{d^2}{d\mu(i) d\mu(j)} [\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) e^{\lambda B(x, \mu_e(\cdot|\mu))}] = 0. \tag{12}$$

Direct calculation leads to

$$\begin{aligned}
& \frac{d^2}{d\mu(i)d\mu(j)}[\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)e^{\lambda B(x,\mu_e(\cdot|\mu)}] \\
= \;& iF^{i-1}(x)\lambda e^{\lambda B(x,\mu_e(\cdot|\mu))}\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) \\
& + jF^{j-1}(x)\lambda e^{\lambda B(x,\mu_e(\cdot|\mu))}\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)e^{\lambda B(x,\mu_e(\cdot|\mu))}\lambda^2\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)e^{\lambda B(x,\mu_e(\cdot|\mu))}\lambda\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu)). \qquad (13)
\end{aligned}$$

Combining (12) and (13), we have

$$\begin{aligned}
& iF^{i-1}(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\lambda\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu)) \\
& = 0. \qquad (14)
\end{aligned}$$

To conduct the concavification procedure, we check the Hessian matrix H of $R(\mu)$ given by (4): $\forall$ $i, j \in \{1, 2, ..., M\}$,

$$\begin{aligned}
\frac{dR(\mu)}{d\mu(i)} \;=\; & \int_0^{\overline{x}} iF^{i-1}(x)f(x)B(x,\mu_e(\cdot|\mu))dx \\
& + \sum_{N=1}^{M}\mu(N)\int_0^{\overline{x}} NF^{N-1}(x)f(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))dx.
\end{aligned}$$

Further calculation, together with (14), results in

$$
\begin{aligned}
\frac{d^2R(\mu)}{d\mu(i)d\mu(j)} &= \int_0^{\overline{x}} iF^{i-1}(x)f(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu))dx \\
&+\int_0^{\overline{x}} jF^{j-1}(x)f(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))dx \\
&+\sum_{N=1}^{M} N\mu(N)\int_0^{\overline{x}} F^{N-1}(x)f(x)\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu))dx \\
&= \int_0^{\overline{x}} [iF^{i-1}(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu)) \\
&+\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu))]f(x)dx \\
&= -\int_0^{\overline{x}} [\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)e^{\lambda B(x,\mu_e(\cdot|\mu))}\lambda\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu))]f(x)dx \\
&= -\lambda\sum_{N=1}^{M} N\mu(N)\int_0^{\overline{x}} [F^{N-1}(x)f(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu))]dx \\
&= -\lambda\sum_{N=1}^{M} N\mu(N)\int_0^{\overline{x}} [F^{N-1}(x)f(x)\beta_i(x,\mu_e(\cdot|\mu))\beta_j(x,\mu_e(\cdot|\mu))]dx,
\end{aligned}
$$

where $\beta_i(x,\mu_e(\cdot|\mu)) := \frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu))$.

Note that, $\forall\alpha = (\alpha_1,...,\alpha_M)^T$, we have

$$
\begin{aligned}
\alpha^T H\alpha &= \sum_{N=1}^{M}\sum_{N=1}^{M}\alpha_i\alpha_j H_{ij} = \sum_{N=1}^{M}\sum_{N=1}^{M}\alpha_i\alpha_j\frac{d^2R(\mu)}{d\mu(i)d\mu(j)} \\
&= \sum_{N=1}^{M}\sum_{N=1}^{M}\alpha_i\alpha_j\{-\lambda\sum_{N=1}^{M}N\mu(N)\int_0^{\overline{x}}F^{N-1}(x)f(x)\beta_i(x,\mu_e(\cdot|\mu))\beta_j(x,\mu_e(\cdot|\mu))dx\} \\
&= -\lambda\sum_{N=1}^{M}N\mu(N)\int_0^{\overline{x}}\{F^{N-1}(x)f(x)\sum_{N=1}^{M}\sum_{N=1}^{M}[\alpha_i\alpha_j\beta_i(x,\mu_e(\cdot|\mu))\beta_j(x,\mu_e(\cdot|\mu))]\}dx \\
&= -\lambda\sum_{N=1}^{M}N\mu(N)\int_0^{\overline{x}}\{F^{N-1}(x)f(x)[\sum_{N=1}^{M}\alpha_i\beta_i(x,\mu_e(\cdot|\mu))]^2\}dx.
\end{aligned}
$$

Since $F^{N-1}(x)f(x) \geq 0$ and $N\mu(N) \geq 0$, we conclude:

If $\lambda > 0$ (risk-averse bidders), H is negative semi-definite and $R(\mu)$ is (weakly) concave, and therefore no disclosure is optimal.

If $\lambda < 0$ (risk-loving bidders), H is positive semi-definite and $R(\mu)$ is (weakly) convex, and therefore full disclosure is optimal.

We now turn to part (iii). Recall that, from Proposition 2, we have

$$\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)B(x,\mu_e(\cdot|\mu)) = \sum_{N=1}^{M} N\mu(N)\int_0^x (N-1)tF^{N-2}(t)f(t)dt.$$

Taking the second order derivative wrt. $\mu(i)$ and $\mu(j)$ on both sides for $\forall i,j \in \{1,2,...,M\}$, we obtain

$$\frac{d^2}{d\mu(i)d\mu(j)}[\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)B(x,\mu_e(\cdot|\mu))] = 0. \tag{15}$$

Direct calculation, together with eqn. (15), leads to

$$\begin{aligned}
& \frac{d^2}{d\mu(i)d\mu(j)}[\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)B(x,\mu_e(\cdot|\mu))] \\
= \;& iF^{i-1}(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu)) \\
= \;& 0.
\end{aligned} \tag{16}$$

Further calculation, together with eqn. (16), leads to

$$\begin{aligned}
\frac{d^2R(\mu)}{d\mu(i)d\mu(j)} \;=\;& \int_0^{\overline{x}} [iF^{i-1}(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)\frac{d}{d\mu(i)}B(x,\mu_e(\cdot|\mu)) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{d^2}{d\mu(i)d\mu(j)}B(x,\mu_e(\cdot|\mu))]f(x)dx \\
=\;& 0.
\end{aligned}$$

Therefore, $H = \mathbf{0}$ and thus $R(\mu)$ is linear in μ. Consequently, any disclosure policy can be optimal.

■

Proof of Theorem 2

Proof. For risk-averse or risk-loving bidders, the ex ante utility of all the potential bidders is

$$EAU := E_\tau E_\mu \int_0^\infty \frac{1}{\lambda}[1 - e^{-\lambda(x - B(x,\mu_e(\cdot|\mu)))}] N F^{N-1}(x) f(x) dx.$$

Recall that, from Proposition 1, we have

$$\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) e^{\lambda B(x,\mu_e(\cdot|\mu))} = \sum_{N=1}^{M} N\mu(N) [\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt],$$

which implies

$$\begin{aligned} & \sum_{N=1}^{M} \mu(N) \int_0^\infty \frac{1}{\lambda} e^{-\lambda(x - B(x,\mu_e(\cdot|\mu)))} N F^{N-1}(x) f(x) dx \\ = & \sum_{N=1}^{M} \mu(N) \int_0^\infty \frac{1}{\lambda} e^{-\lambda x} (\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt) N f(x) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} EAU &= E_\tau E_\mu [\int_0^\infty \frac{1}{\lambda} N F^{N-1}(x) f(x) dx \\ &\quad - \int_0^\infty \frac{1}{\lambda} e^{-\lambda x} (\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt) N f(x) dx] \\ &= E_\tau E_\mu A(N), \end{aligned}$$

where

$$\begin{aligned} A(N) \ : &= \int_0^\infty \frac{1}{\lambda} N F^{N-1}(x) f(x) dx \\ &\quad - \int_0^\infty \frac{1}{\lambda} e^{-\lambda x} (\int_0^x (N-1) e^{\lambda t} F^{N-2}(t) f(t) dt) N f(x) dx]. \end{aligned}$$

Note that $E_\mu A(N)$ is linear in μ, which means the ex ante utility of all the potential bidders is the same across the policies. By symmetry, the ex ante utility of each potential bidder is invariant to the disclosure policy.

Analogously, we can prove the result for the risk-neutral case, where the ex ante utility of all the potential bidders is

$$EAU := E_\tau E_\mu \int_0^\infty [x - B(x, \mu_e(\cdot|\mu)] N F^{N-1}(x) f(x) dx.$$

Recall that, from Proposition 2, we have

$$\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) B(x, \mu(\cdot|s,e)) = \sum_{N=1}^{M} N\mu(N) \int_0^x (N-1) t F^{N-2}(t) f(t) dt,$$

which implies

$$\sum_{N=1}^{M} \mu(N) \int_0^\infty B(x, \mu(\cdot|s,e)) N F^{N-1}(x) f(x) dx = \sum_{N=1}^{M} \mu(N) \int_0^\infty \int_0^x (N-1) t F^{N-2}(t) f(t) dt N f(x) dx.$$

Therefore,

$$\begin{aligned} EAU &= E_\tau E_\mu [\int_0^\infty x N F^{N-1}(x) f(x) dx \\ &\quad - \int_0^\infty \int_0^x (N-1) t F^{N-2}(t) f(t) dt N f(x) dx]. \end{aligned}$$

The linearity in μ implies the ex ante utility of all the potential bidders is the same across all feasible policies. By symmetry, the ex ante utility of each potential bidder is invariant to the disclosure policy. ■

Proof of Lemma 5

Proof. Given the participant i's posterior belief about the number of the participants $\mu_e(\cdot|\mu_s)$ and value $x \in [0, \overline{x}]$, he maximizes his expected utility by choosing his bid b:

$$\max_b U(b, x) := E_{\mu_e(\cdot|\mu_s)}[F^{N-1}(B^{-1}(b))(x - g(b))],$$

i.e.,

$$\max_b \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(B^{-1}(b))(x - g(b))].$$

At a Nash equilibrium:

$$\frac{dU(b)}{db}|_{b=B(x)} = 0,$$

which gives

$$\sum_{N=1}^{M} \mu_e(N|\mu_s)[(N-1)F^{N-2}(x)f(x)(x-g(B(x))\frac{1}{B'(x)} - F^{N-1}(x)g'(B(x))] = 0.$$

That is,

$$g'(B(x))B'(x)\sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x) = \sum_{N=1}^{M} \mu_e(N|\mu_s)[(N-1)F^{N-2}(x)f(x)(x-g(B(x)). \quad (17)$$

Let

$$\begin{aligned} EU(x) \quad &: \quad = EU(B(x)) \\ &= \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(x)(x-g(B(x)))]. \end{aligned} \quad (18)$$

Using (17), we have

$$\begin{aligned} \frac{d}{dx}(EU(x)) &= \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(x)f(x)(x-g(B(x))) \\ &\quad + \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x)(1-g'(B(x))B'(x)) \\ &= \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(x). \end{aligned}$$

Solving this linear differential equation for $EU(x)$, we have

$$EU(x) = K + \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)F^{N-1}(t)dt,$$

where K is a constant, which can be identified from boundary condition $EU(x)|_{x=0} = 0$. We thus have $K = 0$. Therefore,

$$EU(x) = \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(t)dt. \tag{19}$$

From (18) and (19), we have

$$\begin{aligned} & \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(x)(x - g(B(x)))] \\ = & \sum_{N=1}^{M} \mu_e(N|\mu_s) \int_0^x F^{N-1}(t)dt \\ = & \sum_{N=1}^{M} \mu_e(N|\mu_s) e^{-\lambda x}[tF^{N-1}(t)|_0^x - \int_0^x (N-1)F^{N-2}(t)f(t)tdt]. \end{aligned}$$

Thus, the bidding function $B(x, \mu_e(\cdot|\mu_s))$ of an entrant with belief $\mu_e(\cdot|\mu_s)$ and value x satisfies:

$$\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x) g(B(x, \mu_e(\cdot|\mu_s))) = \sum_{N=1}^{M} \mu_e(N|\mu_s)[\int_0^x (N-1)F^{N-2}(t)f(t)tdt],$$

or equivalently

$$B(x, \mu_e(\cdot|\mu_s)) = g^{-1}\left(\frac{\sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{\sum_{N=1}^{M} \mu_e(N|\mu_s) F^{N-1}(x)}\right).$$

■

Proof of Theorem 3

Proof. From Proposition 3, we have

$$\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) g(B(x, \mu_e(\cdot|\mu))) = \sum_{N=1}^{M} N\mu(N)[\int_0^x (N-1)F^{N-2}(t)f(t)tdt].$$

Note that

$$\frac{d^2}{d\mu(i)d\mu(j)}[\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)g(B(x,\mu_e(\cdot|\mu)))] = 0.$$

Direct calculation, together with the above result, gives

$$\begin{aligned}
& \frac{d^2}{d\mu(i)d\mu(j)}[\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)g(B(x,\mu_e(\cdot|\mu)))] \\
= \quad & \frac{d}{d\mu(j)}[iF^{i-1}(x)g(B(x,\mu_e(\cdot|\mu))) + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)g'(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(i)}(B(x,\mu_e(\cdot|\mu)))] \\
= \quad & iF^{i-1}(x)g'(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)g'(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(i)}(B(x,\mu_e(\cdot|\mu))) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)g''(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(i)}(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(j)}(B(x,\mu_e(\cdot|\mu)))] \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)g'(B(x,\mu_e(\cdot|\mu)))\frac{d^2}{d\mu(i)d\mu(j)}(B(x,\mu_e(\cdot|\mu)))] \\
= \quad & 0.
\end{aligned}$$

As $g' > 0$, we have

$$\begin{aligned}
& iF^{i-1}(x)\frac{d}{d\mu(j)}B(x,\mu_e(\cdot|\mu)) + jF^{j-1}(x)\frac{d}{d\mu(i)}(B(x,\mu_e(\cdot|\mu))) \\
& + \sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{d^2}{d\mu(i)d\mu(j)}(B(x,\mu_e(\cdot|\mu)))] \\
= \quad & -\sum_{N=1}^{M} N\mu(N)F^{N-1}(x)\frac{g''(B(x,\mu_e(\cdot|\mu)))}{g'(B(x,\mu_e(\cdot|\mu)))}\frac{d}{d\mu(i)}(B(x,\mu_e(\cdot|\mu)))\frac{d}{d\mu(j)}(B(x,\mu_e(\cdot|\mu)))]. \quad (20)
\end{aligned}$$

Recall that

$$\begin{aligned}
R(\mu) \quad &= \quad E_\mu[\widetilde{R}(\mu,N)] \\
&= \quad \sum_{N=1}^{M} \mu(N)\int_0^{\overline{x}} B(x,\mu_e(\cdot|\mu))NF^{N-1}(x)f(x)dx.
\end{aligned}$$

We next check the Hessian matrix $H = (H_{ij})$, in which element $H_{ij} = \frac{d^2R(\mu)}{d\mu(i)d\mu(j)}$.

$$\frac{dR(\mu)}{d\mu(i)} = \int_0^{\overline{x}} B(x, \mu_e(\cdot|\mu)) iF^{i-1}(x)f(x)dx + \sum_{N=1}^{M} \mu(N) \int_0^{\overline{x}} \frac{d}{d\mu(i)} B(x, \mu_e(\cdot|\mu)) NF^{N-1}(x)f(x)dx.$$

Using (20) and let $\beta(i) := \frac{d}{d\mu(i)}(B(x, \mu_e(\cdot|\mu)))$, we have

$$\begin{aligned}
&\frac{d^2R(\mu)}{d\mu(i)d\mu(j)} \\
= \;& \int_0^{\overline{x}} \frac{d}{d\mu(j)} B(x, \mu_e(\cdot|\mu)) iF^{i-1}(x)f(x)dx \\
&+ \int_0^{\overline{x}} \frac{d}{d\mu(i)} B(x, \mu_e(\cdot|\mu)) jF^{j-1}(x)f(x)dx \\
&+ \sum_{N=1}^{M} \mu(N) \int_0^{\overline{x}} \frac{d^2}{d\mu(i)d\mu(j)} B(x, \mu_e(\cdot|\mu)) NF^{N-1}(x)f(x)dx \\
= \;& \int_0^{\overline{x}} [iF^{i-1}(x) \frac{d}{d\mu(j)} B(x, \mu_e(\cdot|\mu)) + jF^{j-1}(x) \frac{d}{d\mu(i)} B(x, \mu_e(\cdot|\mu)) \\
&+ \sum_{N=1}^{M} N\mu(N)F^{N-1}(x) \frac{d^2}{d\mu(i)d\mu(j)} B(x, \mu_e(\cdot|\mu))] f(x)dx \\
= \;& \int_0^{\overline{x}} [-\sum_{N=1}^{M} N\mu(N)F^{N-1}(x) \frac{g''(B(x, \mu_e(\cdot|\mu)))}{g'(B(x, \mu_e(\cdot|\mu)))} \frac{d}{d\mu(i)}(B(x, \mu_e(\cdot|\mu))) \frac{d}{d\mu(j)}(B(x, \mu_e(\cdot|\mu)))] f(x)dx \\
= \;& -\sum_{N=1}^{M} N\mu(N) \int_0^{\overline{x}} [F^{N-1}(x) \frac{g''(B(x, \mu_e(\cdot|\mu)))}{g'(B(x, \mu_e(\cdot|\mu)))} \beta(i)\beta(j)] f(x)dx.
\end{aligned}$$

Since $N\mu(N) \geq 0$, $F^{N-1}(x)f(x) \geq 0$ and $g' > 0$, we have the following results. If g is convex, then H is negative semi-definite and $R(\mu)$ is weakly concave. Therefore, full concealment is optimal. If g is concave, then H is positive semi-definite and $R(\mu)$ is weakly convex. Therefore, full disclosure is optimal. If g is linear, then $H = 0$ and $R(\mu)$ is linear. Therefore, any disclosure policy yields the same expected revenue.

We next show that a bidder's expected utility does not depend on the disclosure policy. Analogous to the proof of Theorem 2, we could show this result. The ex ante utility of all the potential bidders is

$$EAU := E_\tau E_\mu \int_0^\infty [x - g(B(x, \mu_e(\cdot|\mu))] N F^{N-1}(x) f(x) dx.$$

Recall that, from Proposition 3, we have

$$\sum_{N=1}^{M} N\mu(N) F^{N-1}(x) g(B(x, \mu_e(\cdot|\mu_s))) = \sum_{N=1}^{M} N\mu(N)[\int_0^x (N-1) F^{N-2}(t) f(t) t dt]$$

which implies

$$\sum_{N=1}^{M} \mu(N) \int_0^\infty g(B(x, \mu(\cdot|s,e))) N F^{N-1}(x) f(x) dx = \sum_{N=1}^{M} \mu(N) \int_0^\infty \int_0^x (N-1) t F^{N-2}(t) f(t) dt N f(x) dx.$$

Therefore,

$$EAU = E_\tau \sum_{N=1}^{M} \mu(N)[\int_0^\infty x N F^{N-1}(x) f(x) dx - \int_0^\infty \int_0^x (N-1) t F^{N-2}(t) f(t) dt N f(x) dx].$$

The linearity implies the ex ante utility of all the potential bidders is the same across the policies. By symmetry, the ex ante utility of each potential bidder is invariant to the disclosure policy.

■

7 Appendix B: all-pay auction with nonlinear bidding cost

In this section, we consider an environment that is identical to Section 4 except that we adopt an all-pay auction instead of a first-price auction. The bidders are risk neutral, and their bidding costs are nonlinear. All selected bidders bid simultaneously based on their posterior beliefs about the number of actual bidders. The highest bidder wins and each bidder pays his own bid. We maintain the same set of notations as in Sections 3 and 4.

For an all-pay auction with non-linear bidding cost function $g(\cdot)$, where $g(\cdot) : \mathbf{R}^+ \to \mathbf{R}^+$ is twice continuously differentiable, $g'(\cdot) > 0$ and $g(0) = 0$, we first characterize the unique symmetric pure strategy Bayesian Nash equilibrium. As usual, let $B(\cdot)$ be the corresponding equilibrium bidding function, which is strictly increasing.

Given participant i's posterior belief on the number of participants $\mu_e(\cdot|\mu_s)$ and his value $x \in [0, \overline{x}]$, he maximizes his expected utility by choosing his bid b :

$$\max_b U(b,x) := E_{\mu_e(\cdot|\mu_s)}[F^{N-1}(B^{-1}(b))x] - g(b),$$

i.e.,

$$\max_b \sum_{N=1}^{M} \mu_e(N|\mu_s)[F^{N-1}(B^{-1}(b))x] - g(b).$$

At a Nash equilibrium:

$$\frac{dU(b)}{db}|_{b=B(x)} = 0, \tag{21}$$

which gives

$$\sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(x)f(x)x\frac{1}{B'(x)} - g'(B(x)) = 0,$$

or equivalently,

$$g'(B(x))B'(x) = \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(x)f(x)x. \tag{22}$$

Note that

$$\frac{d}{dx}(g(B(x))) = g'(B(x))B'(x) = \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(x)f(x)x.$$

We have

$$g(B(x)) = K + \int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(t)f(t)tdt,$$

where K is a constant, which can be identified from boundary condition $g(B(x))|_{x=0} = g(0) = 0$. We thus have $K = 0$. Therefore,

$$B(x) = g^{-1}(\int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(t)f(t)tdt). \tag{23}$$

Thus, the bidding function $B(x, \mu_e(\cdot|\mu_s))$ of an entrant with belief $\mu_e(\cdot|\mu_s)$ and value x satisfies:

$$B(x, \mu_e(\cdot|\mu_s)) = g^{-1}(\int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(t)f(t)tdt).$$

Lemma 6 *For an all-pay auction with cost function $g(\cdot)$, if participating bidders hold the same posterior belief $\mu_e(N|\mu_s)$ and Assumption 2 holds, entrants' unique symmetric equilibrium bidding strategy is*

$$B(x, \mu_e(\cdot|\mu_s)) = g^{-1}(\int_0^x \sum_{N=1}^{M} \mu_e(N|\mu_s)(N-1)F^{N-2}(t)f(t)tdt).$$

As in Sections 3 and 4, we assume $\mu_0 \in int(\Delta^M)$, where $\Delta^M = \{\mu \in R^M | \mu(N) \geq 0, \forall N \in \Omega = \{0, 1, ..., M\}$ and $\sum_{N=0}^{M} \mu(N) = 1\}$.

Combining Lemmas 1 and 6, we have the following proposition, which shows that the equilibrium strategy can be alternatively written in terms of belief μ_s.

Proposition 4 *Under Assumption 2, given a disclosure policy π, conditional on a signal realization s, entrants' unique symmetric equilibrium bidding strategy is*

$$B(x, \mu_e(\cdot|\mu_s)) = g^{-1}(\int_0^x \sum_{N=1}^{M} \frac{N\mu_s(N)}{\sum_{N'=1}^{M} N'\mu_s(N')}(N-1)F^{N-2}(t)f(t)tdt). \tag{24}$$

Recall $\mu_s(N) = \mu(N|s) = \frac{\pi(s|N)\mu_0(N)}{\sum_{N'=0}^{M} \pi(s|N')\mu_0(N')}, \forall N \in \{0, 1, ..., M\}$ is a bidder's updated belief upon only the signal realization s (without knowing own entry status).

Let μ denote the belief of a bidder who only observes the public signal but not his entry status. Conditional on N and μ, the sender's payoff is

$$\begin{aligned} \widetilde{R}(\mu, N) \quad &: = R(\mu_e(\cdot|\mu), N) \\ &= N \int_0^{\overline{x}} B(x, \mu_e(\cdot|\mu))dF(x). \end{aligned}$$

Let

$$
\begin{aligned}
R(\mu) &= E_\mu[\widetilde{R}(\mu, N)] \\
&= \sum_{N=1}^{M} \mu(N)N \int_0^{\overline{x}} B(x, \mu_e(\cdot|\mu))dF(x) \\
&= \sum_{N=1}^{M} \mu(N)N \int_0^{\overline{x}} g^{-1}(\int_0^x \sum_{N=1}^{M} \frac{N\mu(N)}{\sum_{N'=1}^{M} N'\mu(N')}(N-1)F^{N-2}(t)f(t)tdt)dF(x) \\
&= \sum_{N=1}^{M} \mu(N)N \int_0^{\overline{x}} g^{-1}(\frac{1}{\sum_{N'=1}^{M} \mu(N')N'} \int_0^x \sum_{N=1}^{M} \mu(N)N(N-1)F^{N-2}(t)f(t)tdt)dF(x) \\
&: = E_\mu N \int_0^{\overline{x}} g^{-1}(\frac{1}{E_\mu N} \sum_{N=1}^{M} \mu(N)N(N-1) \int_0^x F^{N-2}(t)f(t)tdt)dF(x).
\end{aligned}
$$

By Lemma 4, we can write the seller's problem as follows:

$$\max_\tau E_\tau R(\mu)$$

$$s.t. \sum_{\mu \in \text{supp}(\tau)} \tau(\mu)\mu = \mu_0,$$

where $\mu = (\mu(1), \mu(2), ..., \mu(M))$ is of M dimensions as $\sum_{N=1}^{M} \mu(N) = 1 - \mu(0)$, and τ denotes the distribution of posteriors μ. The usual constraint on τ, as specified in Kamenica and Gentzkow (2011), applies.

We solve for the optimal disclosure policy and check the bidders' ex ante utility in our context. Theorem 4 summarizes the results.

Theorem 4 *Under Assumption 2, for an all-pay auction with cost function g, we have*
(i) the optimal disclosure policy is full concealment if g is convex;
(ii) the optimal disclosure policy is full disclosure if g is concave;
(iii) any policy renders the same total expected revenue if g is linear; and

(iv) the ex ante utility of each potential bidder is invariant to the disclosure policy.

Recalling that

$$\begin{aligned} R(\mu) &= E_\mu N \int_0^{\overline{x}} g^{-1}\Big(\frac{1}{E_\mu N}\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt\Big)dF(x) \\ &= E_\mu N \int_0^{\overline{x}} g^{-1}(\gamma(\mu,x))dF(x), \end{aligned}$$

where $\gamma(\mu,x) := \frac{1}{E_\mu N}\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt$.

When g is linear, i.e., $g(x)=cx$ with $c>0$, we have

$$\begin{aligned} R(\mu) &= \int_0^{\overline{x}} \frac{1}{c}\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt)dF(x) \\ &= \frac{1}{c}\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^{\overline{x}}\int_0^x F^{N-2}(t)f(t)tdt)dF(x). \end{aligned}$$

In this case, $R(\mu)$ is linear in μ, and any policy yields the same expected revenue.

When g is nonlinear, we check the Hessian matrix H, where $H_{ij} = \frac{d^2R(\mu)}{d\mu(i)d\mu(j)}$.

$$\begin{aligned} &\frac{dR(\mu)}{d\mu(i)} \\ = \;& i\int_0^{\overline{x}} g^{-1}(\gamma(\mu,x))dF(x) \\ &+E_\mu N\int_0^{\overline{x}}\frac{1}{g'(\gamma(\mu,x))}\frac{i(i-1)\int_0^x F^{i-2}(t)f(t)dt\cdot E_\mu N - i\cdot\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2}dF(x). \end{aligned}$$

$$
\begin{aligned}
& \frac{d^2 R(\mu)}{d\mu(i)d\mu(j)} \\
= \; & i\int_0^{\overline{x}} \frac{1}{g'(\gamma(\mu,x))} \frac{j(j-1)\int_0^x F^{j-2}(t)f(t)tdt \cdot E_\mu N - j\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2} dF(x) \\
& + j\int_0^{\overline{x}} \frac{1}{g'(\gamma(\mu,x))} \frac{i(i-1)\int_0^x F^{i-2}(t)f(t)tdt \cdot E_\mu N - i\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2} dF(x) \\
& + E_\mu N\int_0^{\overline{x}} \frac{1}{g'(\gamma(\mu,x))} \frac{[i(i-1)\int_0^x F^{i-2}(t)f(t)tdt \cdot j - i\cdot j(j-1)\int_0^x F^{j-2}(t)f(t)tdt)][E_\mu N]^2}{[E_\mu N]^4} dF(x) \\
& - E_\mu N\int_0^{\overline{x}} \frac{1}{g'(\gamma(\mu,x))} \frac{i(i-1)\int_0^x F^{i-2}(t)f(t)tdt \cdot E_\mu N \cdot 2jE_\mu N}{[E_\mu N]^4} dF(x) \\
& + E_\mu N\int_0^{\overline{x}} \frac{1}{g'(\gamma(\mu,x))} \frac{i\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt \cdot 2jE_\mu N}{[E_\mu N]^4} dF(x) \\
& + E_\mu N\int_0^{\overline{x}} -\frac{1}{g''(\gamma(\mu,x))} \frac{j(j-1)\int_0^x F^{j-2}(t)f(t)tdt \cdot E_\mu N - j\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2} \\
& \times \frac{i(i-1)\int_0^x F^{i-2}(t)f(t)tdt \cdot E_\mu N - i\cdot\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2} dF(x) \\
: \; & = -E_\mu N\int_0^{\overline{x}} \frac{1}{g''(\gamma(\mu,x))}\beta(i)\beta(j)dF(x),
\end{aligned}
$$

where $\beta(i) := \frac{i(i-1)\int_0^x F^{i-2}(t)f(t)tdt \cdot E_\mu N - i\cdot\sum_{N=1}^{M}\mu(N)N(N-1)\int_0^x F^{N-2}(t)f(t)tdt}{[E_\mu N]^2}$.

Therefore, if g is convex, then H is negative semi-definite and $R(\mu)$ is weakly concave. Therefore, full concealment is optimal. If g is concave, then H is positive semi-definite and $R(\mu)$ is weakly convex. Therefore, full disclosure is optimal.

The ex ante utility of all the potential bidders is

$$EAU := E_\tau E_\mu \int_0^\infty N[xF^{N-1}(x) - g(B(x, \mu_e(\cdot|\mu))]f(x)dx.$$

From Proposition 4, we have,

$$\sum_{N'=1}^{M} N\mu(N)g(B(x, \mu_e(\cdot|\mu))) = \sum_{N=1}^{M} N\mu(N) \int_0^x (N-1)F^{N-2}(t)f(t)tdt.$$

Thus,

$$\begin{aligned} EAU &= E_\tau \sum_{N=1}^{M} \mu(N) \int_0^\infty [NxF^{N-1}(x)f(x) - Ng(B(x, \mu_e(\cdot|\mu))f(x)]dx \\ &= E_\tau \sum_{N=1}^{M} \mu(N)\{\int_0^\infty [NxF^{N-1}(x)f(x) - N\int_0^x (N-1)F^{N-2}(t)f(t)tdtf(x)]dx\}. \end{aligned}$$

The linearity implies the ex ante utility of all the potential bidders is the same across the policies. By symmetry, the ex ante utility of each potential bidder is invariant to the disclosure policy.

References

[1] Robert J Aumann, Michael B Maschler, "Repeated games with incomplete information." *MIT Press*, 1995.

[2] Brocas, Isabelle, and Juan D. Carrillo. "Influence through ignorance." *The RAND Journal of Economics*, 2007, 38(4), 931-947.

[3] Jimmy Chan, Fei Li, and Yun Wang. "Discriminatory persuasion: how to convince a group." 2015, working paper.

[4] Bo Chen, Xiandeng Jiang, and Dmitriy Knyazev. "On disclosure policies in all-pay auctions with stochastic entry." 2015, working paper.

[5] Jeffrey Ely, Alexander Frankel, and Emir Kamenica. "Suspense and surprise." *Journal of Political Economy,* 2015, 123(1), 215-260.

[6] Xin Feng and Jingfeng Lu. "How to disclose the number of participants in contests: a Bayesian persuasion perspective." *Economics Letters*, 2016, 147, 103-107.

[7] Qiang Fu, Qian Jiao, and Jingfeng Lu. "On disclosure policy in contests with stochastic entry." *Public Choice*, 2011, 148(3-4), 419-434.

[8] Sanford Grossman. "The informational role of warranties and private disclosure about product quality." *Journal of Law and Economics,* 1981, 24(3), 461-483.

[9] Emir Kamenica and Matthew Gentzkow. "Bayesian persuasion." *American Economic Review*, 2011, 101(6), 2590-2615.

[10] Fei Li and Peter Norman. "On Bayesian persuasion with multiple senders." 2015, working paper.

[11] Wooyoung Lim and Alexander Matros. "Contests with a stochastic number of players." *Games and Economic Behavior,* 2009, 67(2), 584-597.

[12] Steven Matthews. "Comparing auctions for risk averse buyers: A buyer's point of view." *Econometrica*, 1987, 633-646.

[13] Preston McAfee and John McMillan. "Auctions with a stochastic number of bidders." *Journal of Economic Theory,* 1987, 43(1), 1-19.

[14] Benny Moldovanu and Aner Sela. "The optimal allocation of prizes in contests." *American Economic Review*, 2001, 91(3), 542-558.

[15] Luis Rayo and Ilya Segal. "Optimal information disclosure." *Journal of Political Economy,* 2010, 118(5), 949-987.

[16] Nikolaus Schweizer and Nora Szech. "Optimal revelation of life-changing information," 2012, working Paper (University of Bonn).

[17] Ina Taneva, "Information Design," 2016, working paper.

[18] Yun Wang. "Bayesian persuasion with multiple receivers." 2013, working paper.

[19] Jun Zhang and Junjie Zhou. "Information disclosure in contests: A Bayesian persuasion approach." *The Economic Journal*, 2016, DOI: 10.1111/ecoj.12277.