Ex Ante Efficient Auctions with Bi-Dimensional Private Information on Values and Entry Costs*

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Abstract

This paper studies ex ante efficient auctions in a setting where bidders have bi-dimensional private information on values and entry costs. By resolving several difficulties due to the nature of the bi-dimensional screening problem, we find that a second-price auction with a reserve price equal to the seller's valuation is ex ante efficient, and *any* ex ante efficient auction must be ex post efficient. The crucial point in our approach is to extend the unknown feasible domain of social planner's problem to a suitable domain so that the maximal expected total surplus function has the same maximum in both domains.

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1 Introduction

Endogenous entry of bidders has been widely identified as an important issue in practice. In their study of the U.S. Minerals Management Service "wildcat auctions", Hendricks,

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Pinkse, and Porter (2003) find that less than 25 percent of eligible bidders participate in the auctions held from 1954 to 1970. For the Texas Department of Transportation mowing contract auctions, Li and Zheng (2009) report that only about 28 percent of plan-holders actually submit bids. Similar endogenous entry patterns have been reported by Bajari and Hortacsu (2003) for online auctions, by Athey, Levin, and Seira (2011), Li and Zhang (2010, 2014), and Roberts and Sweeting (2013) for timber auctions, by Krasnokutskaya and Seim (2011) for highway procurement and by Gentry and Stroup (2014) for corporate takeover markets among many others.[1]

Efficient auction design with endogenous entry has attracted significant amount of attention in the literature. The ex ante efficiency of a simple second-price auction with a reserve price equal to seller's value has been confirmed in a number of settings with one-dimensional private information. In particular, Stegeman (1996) shows the efficiency of the said auction for a setting where bidders who are privately informed about their values need to incur *fixed* entry costs to participate in an auction. Levin and Smith (1994) establish the efficiency of the said auction when bidders need to incur a *fixed* cost to discover their values before they participate in an auction. Bergemann and Välimäki (2002) study a setting where the bidders must decide how much to invest to improve the distribution of their values; the efficiency of the said auction also survives their setting.[2]

In this paper, we study a setting where asymmetric bidders have two-dimensional private information on their values and entry costs.[3,4] This setting differs from Stegeman (1996) by assuming that the entry costs are also private information of the bidders. Several interesting questions arise immediately: Do we have the existence of an efficient auction? If yes, what approach should we take to identify it? How does this additional private information of the entry costs affect its design? In particular, should the efficient auction favor the entrants

[1] Fang and Tang (2014) proposed econometric methods for inferring bidders' risk attitudes and estimating entry costs.

[2] The auction design literature with endogenous entry also considered revenue maximization. This literature includes Milgrom (1981), Samuelson (1985), McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Levin and Smith (1994), Ye (2007), Lu (2009, 2010), Celik and Yilankaya (2009), Moreno and Wooders (2011), and Jehiel and Lamy (2015) among many others.

[3] In a setting of two-dimensional private information with symmetric bidders, Xu, Levin and Ye (2013) study the impact of resale on the efficiency of a second price auction.

[4] The entry cost in this paper can also be treated as a communication cost as modelled in Evans (2012).

with higher values net of entry costs?

Due to the additional dimension of private costs, efficient auction design becomes a bi-dimensional screening mechanism design problem. Rochet and Chone (1998) point out the difficulty of characterizing all implementable endogenous entry equilibria for multi-dimensional screening problems. Thus it would be challenging to identify an efficient auction in our two-dimensional setting by adopting a conventional mechanism design approach.

An exploratory but more practical alternative is to adapt the procedures of Levin and Smith (1994), Stegeman (1996) and Bergemann and Välimäki (2002) in their one-dimensional settings to our bi-dimensional private information setting. Their approaches share some common key insights for deriving the efficient auctions. (1) The set of implementable entries is compact. (2) Ignoring the individual rationality and incentive compatibility of the bidders, the *maximal expected total surplus* for each entry is achieved by allocating the object to the entrant (including the seller) with the highest value. (3) Such defined *maximal expected total surplus* function is continuous on the compact set of implementable entries. (4) The existence of an optimal entry (or information acquisition) that maximizes the *maximal expected total surplus* is guaranteed within the set of implementable entries. (5) The optimal entry (or information acquisition) and the associated *maximal expected total surplus* are implemented by the said simple second price auction. While one can conjecture that their result on ex ante efficient auctions in the setting of one-dimensional private information continues to hold in the two-dimensional case, there are several difficulties that are unique to the two-dimensional problem. The central issue is that the domain of the social planner's maximization problem is not well identified in the two-dimensional setting. To the best of our knowledge, our paper is the first to identify this issue and provide a solution by overcoming a number of difficulties as being discussed below.

First, the set of implementable entries is of infinite dimension and unknown for our setting. In addition, its compactness is unclear. For the existence of an optimal entry, we need to work on a carefully identified compact set of entries that may be strictly bigger than the set of implementable entries. This compact set has to be chosen with great care. One minimum requirement is that the set must not miss the entries implemented by efficient auctions if they exist. Since we have no idea about the efficient entries, we would want this set to cover all implementable entries. On the other hand, there are costs for making the set bigger since the risk that the resulted optimal entry goes beyond the implementable set increases. For the purpose of selecting a right compact set, we partially characterize the properties of the implementable entries by adopting a mechanism design approach as in Myerson (1981). We find that any implementable entry must be described through a set of increasing and continuous shutdown curves. Furthermore, the slopes of the shutdown curves must equal the expected winning probabilities of the participating types. This result inspires us to identify a compact set of hypothetical entries that are described by increasing entry curves satisfying a Lipschitz continuity condition. Note that some of those hypothetical entries may not be implementable by any mechanism.

Second, in order to establish the existence of an optimal entry in the identified compact set of hypothetical entries, we need to establish the continuity of the *maximal expected total surplus* as a function on the identified infinite dimensional compact set of hypothetical entries. The required continuity is obtained by applying the Dominated Convergence Theorem. Given this continuity, the classical Extreme Value Theorem implies that an optimal entry within the identified compact set must exist to maximize the *maximal expected total surplus.*

Third, since the identified compact set of hypothetical entries include many non-implementable entries, we need to establish that an optimal entry is indeed implementable and must be implemented by an auction that is ex post efficient.[5] That is, even if an optimal entry exists, whether the optimal entry and the associated *maximal expected total surplus* are implementable remains to be investigated. We establish that every interior threshold type of bidders that fall on the identified optimal shutdown curves must contribute zero to the total surplus. Since the expected payoff of the threshold types is equivalent to their marginal contribution in a second-price auction with a reserve price equal to the seller's valuation, this auction must implement the optimal entry. In addition, this auction always allocates the object to the entrant with the highest value.

The efficiency of the second price auction thus follows. An immediate implication is that any efficient auction must allocate the object to the entrant with the highest value regardless

[5] Here, ex post efficiency means that the object is allocated to the entrant with the highest value.

of the entry costs. This result is somewhat surprising as it implies that an efficient auction cannot favor the entrants with lower entry costs in terms of ex post winning chances. The innovation in our approach of the problem is to extend the unknown domain of implementable entries to a compact domain so that the maximal expected total surplus function has the same maximum in both domains.

Our finding echoes that of Fang and Morris (2006). In a different bi-dimensional private information setting where each bidder observes his own private valuation as well as noisy signals about his opponents' private valuation, Fang and Morris (2006) also establish the efficiency of a second price auction.

The rest of the paper proceeds as follows. Section 2 sets up the auction model with bi-dimensional types (value and entry cost) for bidders. Section 3 provides a partial characterization of implementable entry following a mechanism design approach. Section 4 studies a relaxed design problem where the designer observes buyers' types. The designer who maximizes the total expected surplus can direct buyers' entry, and he can directly allocate the object to any entrant. The eligible entry set is carefully chosen according to the characteristics of implementable entry and the requirement of being compact. Section 5 establishes the ex ante efficient auction for our bi-dimensional setting. Section 6 is the Appendix which contains some technical proofs.

2 The Model

There is one seller who wants to sell one indivisible object to $N(\geq 2)$ potential bidders through an auction. We use $\mathcal{N} = \{1, 2, \cdots, N\}$ to denote the set of all potential bidders. The seller's valuation for the object is v_0, which is public information. Each bidder i's type is described by his private value of the object v_i and private participation cost c_i, where v_i and c_i are bidder i's private information. Here, c_i can be any costs incurred for bid preparation and bidding process, or opportunity costs of bidding. For $i \in \mathcal{N}$, the private information $t_i = (c_i, v_i)$ is distributed on $T_i = [\underline{c}_i, \overline{c}_i] \times [\underline{v}_i, \overline{v}_i]$ following a cumulative distribution function $F_i(\cdot, \cdot)$ with a density function $f_i(\cdot, \cdot) > 0$ on T_i. The distributions $F_i(\cdot, \cdot)$ are public information. We assume that all the t_i are mutually independent across

$i \in \mathcal{N}$. The seller and bidders are risk neutral.

The timing of the auction is as follows.

Time 0: Nature reveals the set of potential bidders N, the seller's value v_0 and the distributions of the private values and participation costs, which are public information. Each bidder i observes his private value v_i and private participation cost c_i, $i \in N$.

Time 1: The seller announces the rule of the auction. We assume that the auction rule does not allocate the object to a nonparticipating bidder, and a nonparticipant does not pay.

Time 2: The bidders simultaneously and confidentially make their participation decisions and announce their bids if they decide to participate. If bidder i participates, he incurs his participation cost c_i. If he does not participate, he simply takes the outside opportunity and receives 0.

Time 3: The payoffs of the seller and the participating bidders are determined according to the announced rule at time 1.

We look for the **ex ante efficient auction** rule that maximizes the expected total surplus of the seller and bidders. Here, the **expected total surplus** equals the difference between the expected winner's value and the expected entry costs of all the entrants.

Based on the semirevelation principle established in Stegeman (1996, Lemma 1), there is no loss of generality to derive the ex ante efficient mechanisms by considering only the truthful direct semirevelation mechanisms. For a truthful direct semirevelation mechanism, every participant reveals truthfully his type, the nonparticipant do not announce their types.

We use a null message $\emptyset$ to denote the signal of a nonparticipant. In a direct semirevelation mechanism, the message space is $M = \prod_{i=1}^{N} M_i$ where $M_i = \{[\underline{c}_i, \overline{c}_i] \times [\underline{v}_i, \overline{v}_i]\} \cup \{\emptyset\}$ is bidder i's message space. The outcome functions announced by the seller accommodate all participation possibilities in the following form: payment function $x_i(\mathbf{m})$ and winning probability function $p_i(\mathbf{m})$ for bidder i, $\forall i \in N$, where $\mathbf{m} = (m_i)_{i=1}^{N}$ is the message vector and $m_i \in M_i$ is the signal of bidder i. We denote the above mechanism by (p, x), where $p = (p_i(\mathbf{m}))_{i=1}^{N}$ and $x = (x_i(\mathbf{m}))_{i=1}^{N}$, and p should satisfy the "feasibility" restrictions: $p_i(\mathbf{m}) \geq 0$, $\forall i \in N$, $\forall \mathbf{m} \in M$, and $\sum_{i=1}^{N} p_i(\mathbf{m}) \leq 1$, $\forall \mathbf{m} \in M$. Nonparticipating bidders have no chance to win the object and their payments to the seller are zero, i.e., $p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0 \ if \ m_i = \emptyset$, $\forall i \in N$.

For the reasons elaborated in the introduction, we will derive the ex ante efficient auction following the procedure described there. A crucial step is to partially characterize the implementable entries.

3 Partial Characterization of Implementable Entries

The following lemma shows that any equilibrium entry must be characterized by a shutdown curve for each potential bidder.[6] Its proof will be given in the Appendix.

Lemma 1: *Any equilibrium entry that is induced by any auction mechanism can be described through N increasing and continuous shutdown curves $C_i(v_i)$ for $i \in \mathcal{N}$: $[\underline{v}_i, \overline{v}_i] \to [\underline{c}_i, \overline{c}_i]$, $\forall i \in \mathcal{N}$. For bidder i with type (c_i, v_i), he participates if $c_i < C_i(v_i)$, and he does not participate if $c_i > C_i(v_i)$.*

Define (v_ℓ^i, v_u^i) to be the interval on which $C_i(v_i)$ falls into $(\underline{c}_i, \overline{c}_i)$, if $C_i(\cdot)$ is not always equal to $\underline{c}_i$ or $\overline{c}_i$.[7] If $C_i(v_i) \equiv \underline{c}_i$, we let $v_\ell^i = v_u^i = \overline{v}_i$; if $C_i(v_i) \equiv \overline{c}_i$, we let $v_\ell^i = v_u^i = \underline{v}_i$.

Lemma 1 specifies the participation for the bidders whose types are not on the shutdown curves. We can specify the participation of bidders whose types are on the shutdown curves as follows:[8] If $v_\ell^i < v_u^i$, we assume that bidder i with types $(C_i(v_i), v_i)$ where $v_i \geq v_\ell^i$ participates, and bidder i with types $(C_i(v_i), v_i)$ where $v_i < v_\ell^i$ does not participate. If $C_i(v_i) \equiv \overline{c}_i$, we assume all types of bidder i participate; If $C_i(v_i) \equiv \underline{c}_i$, we assume no type of bidder i participates. Because the measure of all involved types on the shutdown curves is zero, this specification does not affect the participation and bidding strategies of other types of bidders. More importantly, the expected total surplus is not affected.

Definition 1: *Consider any increasing and continuous function $\sigma_i : [\underline{v}_i, \overline{v}_i] \to [\underline{c}_i, \overline{c}_i]$. Let $\Gamma_p^i(\sigma_i) = \{(c_i, v_i) \in T_i | c_i \leq \sigma_i(v_i)$ and $v_i \geq \hat{v}_i\}$ where $\hat{v}_i = \inf\{v_i \in [\underline{v}_i, \overline{v}_i] : \sigma_i(v_i) > \underline{c}_i\}$, if $\sigma_i(v_i) \not\equiv \underline{c}_i$; otherwise $\Gamma_p^i(\sigma_i) = \emptyset$.*

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[6] Unlike in Stegeman (1996) and Tan and Yilankaya (2006) where potential bidders participate according to their entry thresholds that are points in the single dimensional value spaces of bidders; in our setting potential bidders participate according to their shutdown curves in their two dimensional spaces of values and entry costs.

[7] Note that $C_i(v_i) \equiv c_i \in (\underline{c}_i, \overline{c}_i)$ cannot be an equilibrium shutdown curve.

[8] This specification is supported by Lemma A.1 in the Appendix that establishes the payoffs of bidders whose types are on the shutdown curves if they participate.

Therefore, for a shutdown curve $C_i(\cdot)$, $\Gamma^i_p(C_i),\ i \in \mathcal{N}$ is the set of all participating types of bidder i. Note that $\Gamma^i_p(C_i)$ is empty if $C_i(v_i) \equiv \underline{c}_i$, and $\Gamma^i_p(C_i)$ is T_i if $C_i(v_i) \equiv \bar{c}_i$.

Lemma 1 has shown that any implementable entry must be characterized by a set of continuous and increasing shutdown curves for the bidders. However, it is not true that every set of shutdown curves that are continuous and increasing must correspond to an implementable entry. We next further characterize some useful properties of implementable shutdown curves following a mechanism design approach.

Consider any implementable entry equilibrium described by given shutdown curves $\mathcal{C}$ $= (C_1(\cdot), C_2(\cdot), \cdots, C_N(\cdot))$. For the corresponding shutdown curve $C_i(\cdot)$ of bidder i, define $m_i(t_i) = t_i$ if $t_i \in \Gamma^i_p(C_i)$, and $m_i(t_i) = \emptyset$ if $t_i \notin \Gamma^i_p(C_i)$. Define bidder i's interim payoff when he submits m_i given his private signal t_i:

$$U_i(\mathbf{p}, \mathbf{x}, t_i, m_i) \ = \ \int_{T_{-i}} [v_i p_i(m_i, \mathbf{m}_{-i}(\mathbf{t}_{-i})) - x_i(m_i, \mathbf{m}_{-i}(\mathbf{t}_{-i}))]\mathbf{f}(\mathbf{t_{-i}})\mathbf{dt_{-i}} - c_i \mathbf{1}_{M_i \setminus \{\emptyset\}}(m_i),$$

where $\mathbf{t}_{-i} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_N)$, $T_{-i} = \prod_{j \neq i} T_j$, $\mathbf{m}_{-i}(\mathbf{t}_{-i}))$ denotes the signals of bidders other than i, and $\mathbf{f}(\mathbf{t}_{-i})$ is the density function of $\mathbf{t}_{-i}$. $\mathbf{1}_{M_i \setminus \{\emptyset\}}$ is the indicator function of the set $M_i \setminus \{\emptyset\}$ in M_i. Note that $U_i(p, x, t_i, \emptyset) = 0$.

A direct semirevelation mechanism (p, x) is a truthful one that implements entry of given $\mathcal{C}$ if and only if for all $i \in \mathcal{N}$,

$$U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) \geq 0,\ \forall\ t_i \in \Gamma^i_p(C_i), \tag{1}$$

$$U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) \geq U_i(\mathbf{p}, \mathbf{x}, t_i, t'_i),\ \forall\ t_i, t'_i \in \Gamma^i_p(C_i), \tag{2}$$

$$U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) \geq U_i(\mathbf{p}, \mathbf{x}, t_i, t'_i),\ \forall\ t_i \in \Gamma^i_p(C_i),\ t'_i \notin \Gamma^i_p(C_i), \tag{3}$$

$$U_i(\mathbf{p}, \mathbf{x}, t_i, t'_i) \leq 0,\ \forall\ t_i \notin \Gamma^i_p(C_i), t'_i \in T_i, \tag{4}$$

$$p_i(\mathbf{m}) = x_i(\mathbf{m}) = 0 \ if\ m_i = \emptyset,\ p_i(\mathbf{m}) \geq 0,\ \sum_{j=1}^{N} p_j(\mathbf{m}) \leq 1,\ \forall \mathbf{m} \in \mathcal{M}. \tag{5}$$

(4) guarantees that low-type bidders do not participate while (1) guarantees that high-type bidders do participate. (2) and (3) are IC conditions. In the next proposition (whose proof will be given in the Appendix), we partially characterize an implementable entry by following a mechanism design approach. Note the monotonicity of $C_i(\cdot)$ implies that $C_i(\cdot)$

is differentiable almost everywhere. Nevertheless, the next proposition first establishes the Lipschitz continuity of $C_i(\cdot)$ without relying on its differentiability.

Proposition 1: *For any given implementable shutdown curves $\mathcal{C} = (C_1(\cdot), C_2(\cdot), \cdots, C_N(\cdot))$, we have for each $i \in \mathcal{N}$, C_i satisfies the Lipschitz condition $|C_i(v_i) - C_i(v_i')| \leq |v_i - v_i'|$, $\forall v_i, v_i' \in (\underline{v}_i, \overline{v}_i)$. Moreover, $C_i(\cdot)$ is differentiable almost everywhere, and the derivative $C_i'(v_i)$ (when it exists) gives entrant i's expected winning probability regardless of his entry costs if his value is $v_i \in (v_l^i, v_u^i)$.*

4 A Relaxed Problem

Note that Proposition 1 above does not provide a full characterization of implementable shutdown curves. As a matter of fact, it is well expected that to fully pin down the set of all implementable entries in our two-dimensional setting is very challenging if not impossible. For this reason, it is not realistic to study ex ante efficient auctions via the set of implementable shutdown curves. On the other hand, based on Proposition 1, we can restrict our attention to the increasing and continuous shutdown curves which satisfy the Lipschitz condition when searching for the ex ante efficient entries. This observation leads us to consider N increasing and continuous functions from $[\underline{v}_i, \overline{v}_i]$ to $[\underline{c}_i, \overline{c}_i]$ satisfying the Lipschitz condition for $i \in \mathcal{N}$ as a hypothetical entry. We can thus study ex ante efficient auctions by considering a relaxed optimization problem.

Definition 2: *Let $\Omega_i = \{\omega_i(\cdot) : [\underline{v}_i, \overline{v}_i] \to [\underline{c}_i, \overline{c}_i] \mid \omega_i(\cdot)$ is increasing and $|\omega_i(v_i) - \omega_i(v_i')| \leq |v_i - v_i'|, \ \forall v_i, v_i' \in [\underline{v}_i, \overline{v}_i]\}$. Let d_i be the usual metric on the space $\mathcal{F}_i$ of continuous functions on $[\underline{v}_i, \overline{v}_i]$, i.e., $d_i(\omega_i, \omega_i') = \max_{v_i \in [\underline{v}_i, \overline{v}_i]}\{|\omega_i(v_i) - \omega_i'(v_i)|\}$ for $\omega_i, \omega_i' \in \mathcal{F}_i$. Let $\Omega = \prod_{i=1}^N \Omega_i$ with a metric $d(\omega, \omega') = \sum_{i=1}^N d_i(\omega_i, \omega_i')$ for $\omega = (\omega_1, \ldots, \omega_N)$, $\omega' = (\omega_1', \ldots, \omega_N') \in \Omega$.*

Lemma 2: *The space (Ω, d) is compact.*

Proof: Fix any $i \in \mathcal{N}$. Any function in Ω_i is also bounded by $\overline{c}_i$. By the Lipschitz condition, Ω_i is an equicontinuous family of real valued functions on the interval $[\underline{v}_i, \overline{v}_i]$. The classical Ascoli-Arzelá Theorem as on page 208 of Royden and Fitzpatrick (2010) implies that the closure of Ω_i in $\mathcal{F}_i$ is compact. Since the monotonicity, continuity and Lipschitz conditions are preserved under the uniform limit as defined by the metric d_i, Ω_i is a closed subset of $\mathcal{F}_i$. Hence Ω_i is compact. The compactness of (Ω, d) thus follows. $\square$

We consider a relaxed problem where the designer can observe all buyers' types (c_i, v_i), and he can direct any buyer i to participate if and only if $t_i \in \Gamma_p^i(\omega_i)$ according to any N curves $\omega = (\omega_i(\cdot)) \in \Omega$ and allocate the object to any entrant (including the seller). The designer chooses the optimal ω and allocation rule to maximize the total expected surplus. Clearly, for any ω the designer should allocate the object to the entrant with the highest value. In other words, the above allocation rule renders the *maximal expected total surplus* (METS) for any given entry corresponding to ω. Denote the METS corresponding to ω by $S(\omega)$. Here, we would like to emphasize that the participation and incentive compatibility conditions of bidders are ignored.

Let $\mathbf{t}= (t_i)_{i=1}^N$ and $v_h(\mathbf{t};\boldsymbol{\omega})$ to denote the highest value of the seller and all participating bidders for given ω. We thus have

$$v_h(\mathbf{t};\boldsymbol{\omega}) = \begin{cases} \max\{v_0, \max_{\{\forall j;(c_j,v_j)\in\Gamma_p^j(w_j)\}} v_j\}, & \text{if} \quad \{j \in \mathcal{N} | (c_j, v_j) \in \Gamma_p^j(\omega_j)\} \neq \emptyset, \\ v_0, & \text{if} \quad \{j \in \mathcal{N} | (c_j, v_j) \in \Gamma_p^j(\omega_j)\} = \emptyset. \end{cases}$$

The maximal expected total surplus $S(\omega)$ can be written as

$$S(\omega) = Ev_h(\omega) - \sum_{i\in\mathcal{N}} \int_{t_i=(c_i,v_i)\in\Gamma_p^i(w_i)} c_i f_i(t_i) dt_i.$$

The following lemma shows the continuity of $S(\cdot)$ on Ω.

Lemma 3: *The maximal expected total surplus $S(\cdot)$ is continuous on Ω.*

It is thus obvious from Lemmas 2 and 3 that there exists a $\omega^* \in \Omega$ that maximizes $S(\omega)$. In the next lemma, we characterize the properties of the optimal ω^*, which provide the key insights for us to solve our bi-dimensional screening problem. The proofs for both Lemmas 3 and 4 will be given in the Appendix.

Lemma 4: *There must exist $\omega^* = (\omega_i^*) \in \Omega$ that maximizes $S(\cdot)$ within Ω. Bidder i with types $t_i \in T_i$ where $t_i \in (\notin)\Gamma_p^i(\omega_i^*)$ must contribute nonnegatively (nonpositively) to $S(\omega^*)$ if he participates, given that any other bidder $j \in \mathcal{N}\setminus\{i\}$ participates if and only if $t_j \in \Gamma_p^j(\omega_j^*)$*

and the object is allocated to the participant (including the seller) with the highest valuation.

5 Ex Ante Efficient Auction

When we characterize ω^* in the previous section, we ignore the full implementability of the hypothetical entries in Ω. It remains a key issue whether the optimal ω^* characterized in Lemma 4 is indeed implementable. For this purpose, we need to show that it is implementable by an auction which allocates the object to the participant with the highest valuation. The derivation is based on the insight that the necessary conditions provided in Lemma 4 are sufficient conditions for a second-price auction to implement ω^*.

Theorem 1: *The second-price auction with a reserve price equal to seller's valuation and zero entry fee has an entry and bidding equilibrium that maximizes the expected total surplus of all seller and bidders.*

Proof: Note that the contribution of type t_i to $S(\omega^*)$ can be alternatively interpreted as the expected payoffs of the bidder i of type t_i in this second-price auction, if other bidder $j(\neq i)$ participates if and only if $t_j \in \Gamma_p^j(\omega_j^*)$ and bids their true values when participating. It is thus an entry equilibrium in the second-price auction that bidder i participates if and only if $t_i \in \Gamma_p^i(\omega_i^*)$. This auction induces entry ω^* and truthful bidding of entrants, thus it achieves $S(\omega^*)$. $\square$

The procedure of showing the ex ante efficient auction illustrates clearly the intuition behind Theorem 1. A set of necessary conditions for the hypothetical optimal entry $\omega^* \in \Omega$ are sufficient for it to be implemented through a second-price auction with a reserve price equal to seller's valuation and zero entry fee, which always awards the object to the participant with the highest value.

Cao et al. (2015) reveal that in general there exist multiple entry equilibria in second price auctions with bi-dimensional private information of values and entry costs. This means that the ex ante efficient auction identified in Theorem 1 in general has inefficient entry equilibria other than the efficient one. Nevertheless, Cao et al. (2015) establish sufficient conditions for the uniqueness of entry equilibrium for environments with two potential bidders.

We define ex post efficiency by allocation of the object to the entrant with the highest value. The following result entails.

Corollary 1: *Any ex ante efficient auction must be ex post efficient, i.e., it always allocates the object to the participant with the highest valuation regardless of the entry cost profile.*

Proof: Denote the efficient entry implemented by an ex ante efficient auction $\mathcal{A}$ by $\mathcal{C}$. According to Proposition 1, we have $\mathcal{C} \in \Omega$. For $\mathcal{C}$, the METS is attained if the object is assigned to the participant with the highest valuation. This METS cannot be higher than that of the Theorem 1 ex ante efficient auction by the definition of ω^*. However, if auction $\mathcal{A}$ fails to allocate the object to the bidder with the highest valuation with a positive probability, then the total expected surplus achieved must be strictly lower than that of the ex ante efficient auction of Theorem 1. However, this means that auction $\mathcal{A}$ cannot be ante efficient. Therefore, any ex ante efficient auction must be ex post efficient. $\square$

Corollary 1 implies that an efficient auction cannot favor the entrants with lower entry costs in terms of ex post winning chances. This is somewhat surprising since such an allocation rule could provide higher incentive to more cost-efficient buyers to participate, which could possibly increase the ex ante efficiency by reducing entry costs of the participants.

Moreover, Corollary 1 means that the ex ante efficiency of the second price auction of Theorem 1 in general can not be extended to a first price or all pay auction when bidders are asymmetric since in these auctions ex post efficiency is not guaranteed.

6 Appendix

Proof of Lemma 1: *First*, we show the existence of shutdown curves. For any entry equilibrium induced by any given auction rule, consider any type $(c_i, v_i) \in T_i$. If this type of bidder i participates with a positive probability, bidder i with types (c_i', v_i) where $c_i' < c_i$ must participate with probability 1. The arguments are the following. If bidder i with (c_i', v_i) where $c_i' < c_i$ participates with probability 1 and mimics the bidding strategy of type (c_i, v_i), he gets strictly positive expected payoff since his entry cost is lower. This implies that bidder i with (c_i', v_i) must gain strictly positive payoff when he participates and bids optimally. Thus bidder i with (c_i', v_i) participates with probability 1. Equivalently, if bidder i with (c_i, v_i) participates with probability 0, bidder i with types (c_i', v_i) where $c_i' > c_i$ must participate with probability 0. Based on this observation, for each $v_i \in [\underline{v}_i, \overline{v}_i]$, we have a critical value $C_i(v_i) \in [\underline{c}_i, \overline{c}_i]$ so that bidder i with types (c_i, v_i) where $c_i < C_i(v_i)$ must participate with probability 1, and bidder i with types (c_i, v_i) where $c_i > C_i(v_i)$ must not participate. Note that there is no possibility of stochastic participation unless for types $(v_i, C_i(v_i)), \forall v_i \in [\underline{v}_i, \overline{v}_i]$.

Second, we consider the monotonicity of these shutdown curves. We claim that $C_i(v_i) \geq C_i(v_i')$, if $v_i > v_i'$. We show this by contradiction. Suppose $C_i(v_i) < C_i(v_i')$ for $\overline{v}_i \geq v_i > v_i' \geq \underline{v}_i$. Consider bidder i with type (c_i, v_i) where $c_i \in (C_i(v_i), C_i(v_i'))$. If he participates and mimics the strategy of type (c_i, v_i'), his expected payoff is at least equal to that of type (c_i, v_i'), which is strictly positive. This leads to that bidder i with type (c_i, v_i) must participate with probability 1. This conflicts with the assumption that bidder i with type (c_i, v_i) where $c_i \in (C_i(v_i), C_i(v_i'))$ does not participate.

Third, we show that $C_i(v_i)$ is continuous. Suppose $C_i(\cdot)$ is not continuous. Without loss of generality, we assume that $\lim_{v \to v_i-} C_i(v) < C_i(v_i)$ for some $v_i \in (\underline{v}_i, \overline{v}_i)$. Then we must have $\lim_{v \to v_i-} C_i(v) < C_i(v_i)$ since $C_i(\cdot)$ is increasing. Note that $C_i(\cdot)$ is a bounded increasing function, so we have $\lim_{v \to v_i-} C_i(v)$ exists. Consider bidder i with type $(\tilde{c}_i, v_i)$ where $\tilde{c}_i \in (\lim_{v \to v_i-} C_i(v), C_i(v_i))$. The expected payoff of bidder i with type $(\tilde{c}_i, v_i)$ must be strictly positive. Then bidder i with type $(\tilde{c}_i, \tilde{v}_i)$ where $\tilde{v}_i$ is slightly smaller than v_i also gets strictly positive expected payoff if he mimics the strategy of type $(\tilde{c}_i, v_i)$. This result conflicts with the assumption that $(\tilde{c}_i, \tilde{v}_i)$ does not participate. □

Lemma A.1: *Consider shutdown curve $C_i(\cdot)$ for any bidder $i \in \mathcal{N}$. (i) If $v_\ell^i < v_u^i$, the expected* *payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in [v_\ell^i, v_u^i]$ is exactly zero if he participates; the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in [\underline{v}_i, v_\ell^i)$ is no bigger than zero if he participates; the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in (v_u^i, \overline{v}_i]$ is no smaller than zero if he participates.*

(ii) If $v_\ell^i = v_u^i = \overline{v}_i$ (i.e., $C_i(v_i) \equiv \underline{c}_i$), the expected payoff of bidder i with types $(\underline{c}_i, v_i)$ where $v_i \in [\underline{v}_i, \overline{v}_i]$ is no bigger than zero if he participates.

(iii) If $v_\ell^i = v_u^i = \underline{v}_i$ (i.e., $C_i(v_i) \equiv \overline{c}_i$), the expected payoff of bidder i with types $(\overline{c}_i, v_i)$ where $v_i \in [\underline{v}_i, \overline{v}_i]$ is no smaller than zero if he participates.

Proof of Lemma A.1: We first consider the case where $v_\ell^i < v_u^i$. We show that the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in (v_\ell^i, v_u^i)$ is exactly zero if he participates. On one hand, it can not be bigger than zero, otherwise, bidder i with type $(C_i(v_i) + \epsilon, v_i)$ will participate and mimic the strategy of $(C_i(v_i), v_i)$ to get strictly positive expected payoff. This conflicts with the definition of $C_i(v_i)$. On the other hand, it can not be smaller than zero. If he participates and mimics the strategy of type $(C_i(v_i) - \epsilon, v_i)$, he will at least get $-\epsilon$ where ϵ can be smaller than any positive number. This conflicts with the fact that his best expected payoff is negative if he participates. At (c_ℓ^i, v_ℓ^i), the expected payoff of bidder i can not be bigger than zero. Otherwise, bidder i with type $(c_\ell^i + \epsilon, v_\ell^i)$ will participate and mimic the strategy of (c_ℓ^i, v_ℓ^i) to get strictly positive expected payoff. At (c_ℓ^i, v_ℓ^i), the expected payoff of bidder i can also not be smaller than zero. This is because if he participates and mimics the strategy of type $(c_\ell^i, v_\ell^i + \epsilon)$ then he will at least get $-\epsilon$ where ϵ can be smaller than any positive number. Thus the expected payoff of bidder i with type (c_ℓ^i, v_ℓ^i) is exactly zero if he participates. The same result holds for type (c_u^i, v_u^i). Similar arguments lead to that the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in [\underline{v}_i, v_\ell^i)$ is no bigger than zero if he participates, and the expected payoff of bidder i with types $(C_i(v_i), v_i)$ where $v_i \in (v_u^i, \overline{v}_i]$ is no smaller than zero if he participates.

The results for the two cases where $v_\ell^i = v_u^i$ can be similarly shown. □

Proof of Proposition 1: Define conditional expected winning probability and payments

$$Q_i(\mathbf{p}, t_i) = \int_{T_{-i}} p_i(m_1(t_1), \cdots, m_i(t_i), \cdots, m_N(t_N)) f(\mathbf{t}_{-i}) d\mathbf{t}_{-i},$$
$$X_i(\mathbf{p}, t_i) = \int_{T_{-i}} x_i(m_1(t_1), \cdots, m_i(t_i), \cdots, m_N(t_N)) f(\mathbf{t}_{-i}) d\mathbf{t}_{-i}.$$

We first establish the Lipschitz continuity of the shutdown curves. For this purpose, we only need to show that $\forall\ v_i > v_i' \geq v_\ell^i$, we have $C_i(v_i) - C_i(v_i') \leq v_i - v_i'$. Let $c_i = C_i(v_i)$ and $c_i' = C_i(v_i')$. Let $t_i = (c_i, v_i), t_i' = (c_i', v_i')$. Incentive compatibility implies that

$$\begin{aligned}
& U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) \\
= \ & v_i Q_i(\mathbf{p}, t_i) - X_i(\mathbf{p}, t_i) - c_i \\
\geq \ & U_i(\mathbf{p}, \mathbf{x}, t_i, t_i') \\
= \ & v_i Q_i(\mathbf{p}, t_i') - X_i(\mathbf{p}, t_i') - c_i \\
= \ & (v_i' + v_i - v_i') Q_i(\mathbf{p}, t_i') - X_i(\mathbf{p}, t_i') - c_i' + c_i' - c_i \\
= \ & U_i(\mathbf{p}, \mathbf{x}, t_i', t_i') + (v_i - v_i') Q_i(\mathbf{p}, t_i') + (c_i' - c_i).
\end{aligned}$$

Note that $U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) = U_i(\mathbf{p}, \mathbf{x}, t_i', t_i') = 0$ as these two types are indifferent between participating and not participating. Therefore, we have $0 \geq (v_i - v_i') Q_i(\mathbf{p}, t_i') + c_i' - c_i$, which leads to $c_i - c_i' \leq (v_i - v_i') Q_i(\mathbf{p}, t_i') \leq v_i - v_i'$. Hence, the shutdown curve is Lipschitz continuous when $v \in [v_l^i, v_u^i]$, which further implies that $C_i(\cdot)$ is Lipschitz continuous when $v_i \in [\underline{v}_i, \overline{v}_i]$.

Note that the monotonicity of $C_i(\cdot)$ implies that $C_i(\cdot)$ is differentiable almost everywhere. We next show that the derivative $C_i'(v_i)$ (when it exists) of the shutdown curve C_i at $v_i \in (v_\ell^i, v_u^i)$ equals exactly the expected winning probability of the participant i with value v_i regardless of his entry costs. Using $U_i(\mathbf{p}, \mathbf{x}, t_i, t_i)$ and $Q_i(\mathbf{p}, t_i)$, we can reinterpret conditions (1) to (5), when $C_i(\cdot) \neq \underline{c}_i$ and $C_i(\cdot) \neq \overline{c}_i$, for all $i \in \mathcal{N}$. If a direct mechanism $(\mathbf{p}, \mathbf{x})$ is a truthful one that implements entry $\mathcal{C}$, then for all $i \in \mathcal{N}$, the following conditions hold. Note that these conditions hold almost everywhere.

$$Q_i(\mathbf{p}, t_i) \geq Q_i(\mathbf{p}, t_i'),\ \forall\ t_i = (c_i, v_i), t_i' = (c_i, v_i') \in \Gamma_p^i(C_i), v_i \geq v_i', \forall\ i \in \mathcal{N}, \tag{A.1}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}, t_i, t_i)}{\partial v_i} = Q_i(\mathbf{p}, t_i),\ \forall t_i = (c_i, v_i) \in \Gamma_p^i(C_i),\ \forall i \in \mathcal{N}, \tag{A.2}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}, t_i, t_i)}{\partial v_i} = C_i'(v_i),\ \forall t_i = (c_i, v_i) \in \Gamma_p^i(C_i),\ \textit{where}\ v_i \in [v_\ell^i, v_u^i],\ \forall\ i \in \mathcal{N}, \tag{A.3}$$

$$\frac{\partial U_i(\mathbf{p}, \mathbf{x}, t_i, t_i)}{\partial c_i} = -1,\ \forall t_i = (c_i, v_i) \in \Gamma_p^i(C_i),\ \forall i \in \mathcal{N}, \tag{A.4}$$

$$U_i(\mathbf{p}, \mathbf{x}, t_i, t_i) = 0,\ \forall\ t_i = (C_i(v_i), v_i),\ \textit{where}\ v_i \in [v_\ell^i, v_u^i],\ \forall\ i \in \mathcal{N}. \tag{A.5}$$

(A.5) is from Lemma A.1(i). Similar procedure as in Myerson (1981) leads to (A.1), (A.2) to (A.4) by considering single dimensional deviations of bidder i.[9] Take $v_\ell^i < v_i < v_i' < v_u^i$, and let $c_i = C_i(v_i)$, $c_i = C_i(v_i')$. Note $U_i(\mathbf{p}, \mathbf{x}, (c_i, v_i), (c_i, v_i)) = 0$ and $U_i(\mathbf{p}, \mathbf{x}, (c_i', v_i'), (c_i', v_i')) = 0$. Therefore,

$$[U_i(\mathbf{p}, \mathbf{x}, (c_i', v_i'), (c_i', v_i')) - U_i(\mathbf{p}, \mathbf{x}, (c_i, v_i'), (c_i, v_i'))]$$
$$+[U_i(\mathbf{p}, \mathbf{x}, (c_i, v_i'), (c_i, v_i')) - U_i(\mathbf{p}, \mathbf{x}, (c_i, v_i), (c_i, v_i))] = 0.$$

When $v_i' \to v_i$, we have $c_i' \to c_i$. (A.2) and (A.4) thus lead to (A.3). (A.3) and (A.2) give $C_i'(v_i) = Q_i(\mathbf{p}, t_i),\ \forall t_i = (c_i, v_i) \in \Gamma_p^i(C_i),\ \textit{where}\ v_i \in (v_\ell^i, v_u^i),\ \forall\ i \in \mathcal{N}$.

From (A.2) and (A.3), we have $C_i'(v_i) = Q_i(\mathbf{p}, t_i) \in [0,1]$, $\forall v_i \in (v_\ell^i, v_u^i)$ if $C_i(\cdot) \neq \underline{c}_i$ and $C_i(\cdot) \neq \bar{c}_i$, $\forall i \in \mathcal{N}$. $\square$

Proof of Lemma 3: Let $\{\omega^n\}_{n=0}^\infty$ to be a sequence of curve sets in Ω such that for each $i \in \mathcal{N}$, $\sup_{x \in [\underline{v}_i, \bar{v}_i]} |\omega_i^n(x) - \omega_i^0(x)|$ converges to zero as $n \to \infty$. We need to prove that $\lim_{n\to\infty} S(\omega^n) = S(\omega^0)$. For notational simplicity, we denote $\Gamma_p^j(\omega_j^n)$ by D_{nj}, and $T_j \setminus D_{nj}$ by D_{nj}^c. For a set K, let $\mathbf{1}_K$ denote the indicator function of the set K; that is $\mathbf{1}_K$ is 1 on K and 0 outside K. We can rewrite $v_h(\mathbf{t}; \boldsymbol{\omega})$ as follows:

$$v_h(\mathbf{t}; \omega^n) = \sum_{I \subseteq \mathcal{N}} \max_{j \in I \cup \{0\}} v_j \prod_{j \in I} \mathbf{1}_{D_{nj}}(t_j) \prod_{k \in \mathcal{N} \setminus I} \mathbf{1}_{D_{nk}^c}(t_k),$$

where I is a subset of $\mathcal{N}$, including the empty set. Hence, it follows that

$$E_{\mathbf{t}} v_h(\mathbf{t}; \omega^n) = \sum_{I \subseteq \mathcal{N}} \int_{(t_1, \ldots, t_N) \in T_1 \times \cdots \times T_N} \max_{j \in I \cup \{0\}} \{v_j\} \prod_{j \in I} \mathbf{1}_{D_{nj}}(t_j) \prod_{k \in \mathcal{N} \setminus I} \mathbf{1}_{D_{nk}^c}(t_k) \prod_{i=1}^N f_i(t_i) dt_1 \ldots dt_N.$$

For each $i = 1, \ldots, N$, choose $t_i = (c_i, v_i)$ not on the curve ω_i^0. We have $\omega_i^0(v_i) > c_i$ or $\omega_i^0(v_i) < c_i$. By the uniform convergence property of $\{\omega^n\}_{n=1}^\infty$, we obtain that $\omega_i^n(v_i) > c_i$ or $\omega_i^n(v_i) < c_i$ for n large enough, which also implies that $\lim_{n\to\infty} \mathbf{1}_{D_{ni}}(t_i) = \mathbf{1}_{D_{0i}}(t_i)$ and $\lim_{n\to\infty} \mathbf{1}_{D_{ni}^c}(t_i) = \mathbf{1}_{D_{0i}^c}(t_i)$. For each fixed subset I of $\mathcal{N}$, the sequence of functions

$$\max_{j \in I \cup \{0\}} \{v_j\} \prod_{j \in I} \mathbf{1}_{D_{nj}}(t_j) \prod_{k \in \mathcal{N} \setminus I} \mathbf{1}_{D_{nk}^c}(t_k) \prod_{i=1}^N f_i(t_i),$$

[9] Detailed proofs are available from the authors upon request.

is dominated by the integrable function $\max_{j\in\mathcal{N}\cup\{0\}}\{\bar{v}_j\}\prod_{i=1}^{N} f_i(t_i)$ where $\overline{v}_0 = v_0$ and converges to

$$\max_{j\in I\cup\{0\}}\{v_j\}\prod_{j\in I}\mathbf{1}_{D_{0j}}(t_j)\prod_{k\in\mathcal{N}\setminus I}\mathbf{1}_{D_{0k}^c}(t_k)\prod_{i=1}^{N} f_i(t_i), \forall I,$$

except on the null set $\prod_{i\in\mathcal{N}} graph(\omega_i^0)$. By the Dominated Convergence Theorem (see, for example, page 376 of Royden and Fitzpatrick (2010)),

$$\begin{aligned}
&\lim_{n\to\infty} E_{\mathbf{t}} v_h(\mathbf{t};\omega^n)\\
=\ &\sum_{I\subseteq\mathcal{N}}\int_{(t_1,\ldots,t_N)\in T_1\times\cdots\times T_N}\max_{j\in I\cup\{0\}}\{v_j\}\prod_{j\in I}\mathbf{1}_{D_{0j}}(t_j)\prod_{k\in\mathcal{N}\setminus I}\mathbf{1}_{D_{0k}^c}(t_k)\prod_{i=1}^{N} f_i(t_i)dt_1\ldots dt_N\\
=\ &E_{\mathbf{t}} v_h(\mathbf{t};\omega^0).
\end{aligned}$$

We now look at the second term in $S(\omega^n)$.

$$-\sum_{i\in\mathcal{N}}\int_{t_i=(c_i,v_i)\in\Gamma_p^i(w_i^n)} c_i f_i(t_i)dt_i = -\int_{(t_1,\ldots,t_N)\in T_1\times\cdots\times T_N}\sum_{i\in\mathcal{N}}[c_i\mathbf{1}_{D_{ni}}(t_i)]\prod_{i=1}^{N} f_i(t_i)dt_1\ldots dt_N.$$

$\sum_{i\in\mathcal{N}}[c_i\mathbf{1}_{D_{ni}}(t_i)]\prod_{i=1}^{N} f_i(t_i)$ is dominated by the integrable function $\sum_{i\in\mathcal{N}}\bar{c}_i\prod_{i=1}^{N} f_i(t_i)$ and converges to $\sum_{i\in\mathcal{N}}[c_i\mathbf{1}_{D_{n0}}(t_i)]\prod_{i=1}^{N} f_i(t_i)$ except on the null set $\prod_{i\in\mathcal{N}} graph(\omega_i^0)$.

By the same Dominated Convergence Theorem,

$$\lim_{n\to\infty}\sum_{i\in\mathcal{N}}\int_{t_i=(c_i,v_i)\in\Gamma_p^i(w_i^n)} c_i f_i(t_i)dt_i = \sum_{i\in\mathcal{N}}\int_{t_i=(c_i,v_i)\in\Gamma_p^i(w_i^0)} c_i f_i(t_i)dt_i.$$

Therefore, $\lim_{n\to\infty} S(\omega^n) = S(\omega^0)$. $\square$

Proof of Lemma 4: As $S(\cdot)$ is a continuous function defined on a compact space Ω, according to the classical Extreme Value Theorem (see page 200 of Royden and Fitzpatrick (2010)), there must exist $\omega^* \in \Omega$ that maximizes $S(\cdot)$ within Ω.

For any given N curves $\omega \in \Omega$, define the following random variables: $\forall i \in \mathcal{N}$,

$$\tilde{v}_i(\mathbf{t}_{-i};\omega_{-i}) = \begin{cases} \max\{v_0, \max_{\{\forall j \neq i;(c_j,v_j)\in\Gamma^j_p(\omega_j)\}} v_j\}, & \text{if} \quad \{j \in \mathcal{N}\setminus\{i\}|(c_j,v_j)\in\Gamma^j_p(\omega_j)\} \neq \emptyset, \\ v_0, & \text{if} \quad \{j \in \mathcal{N}\setminus\{i\}|(c_j,v_j)\in\Gamma^j_p(\omega_j)\} = \emptyset. \end{cases}$$

$\tilde{v}_i(\mathbf{t}_{-i};\omega_{-i})$ thus is the highest value of the seller and all participating bidders except i. If $\{j \in \mathcal{N}\setminus\{i\}|\Gamma^j_p(\omega_j) \neq \emptyset\} = \emptyset$, $\tilde{v}_i \equiv v_0$. Denote the cumulative distribution function of $\tilde{v}_i$ by $\tilde{G}_i(\cdot;\omega_{-i})$, where $\omega_{-i} = (\omega_1,\cdots,\omega_{i-1},\omega_{i+1},\cdots,\omega_N)$.

For simplicity, let $D_j = \Gamma^j_p(\omega_j)$ and $D^c_j = T_j \setminus D_j$. $\tilde{v}_i$ can be rewritten as:

$$\tilde{v}_i(\mathbf{t}_{-i};\omega_{-i}) = \sum_{I\subseteq\mathcal{N}\setminus\{i\}} \max_{j\in I\cup\{0\}} v_j \prod_{j\in I} \mathbf{1}_{D_j}(t_j) \prod_{k\in\mathcal{N}\setminus\{i\}\setminus I} \mathbf{1}_{D^c_k}(t_k).$$

We extend f_j so that $f_j(t_j) = 0$ when $t_j \in \mathbb{R}^2\setminus T_j$. Thus F_j can be defined on $\mathbb{R}^2$ as well. Let μ_j be the probability measure on $\mathbb{R}^2$ induced by F_j. Fix any $v \geq v_0$. It is clear that $\max_{j\in I\cup\{0\}} v_j \leq v$ if and only if $v_j \leq v$ for each $j \in I$. In particular, $\max_{j\in I\cup\{0\}} v_j \leq v_0$ if and only if $v_j \leq v_0$ for each $j \in I$. By the independence assumption, we know that the probability of $\tilde{v}_i \leq v$ is

$$\sum_{I\subseteq\mathcal{N}\setminus\{i\}} \prod_{j\in I} \int_{-\infty}^{v} \int_{-\infty}^{\omega_j(v_j)} f_j(c_j,v_j)dc_jdv_j \prod_{k\in\mathcal{N}\setminus\{i\}\setminus I} \mu_k(D^c_k).$$

Thus, the probability of $\tilde{v}_i = v_0$ is

$$\tilde{G}_i(v_0;\omega_{-i}) = \sum_{I\subseteq\mathcal{N}\setminus\{i\}} \prod_{j\in I} \int_{-\infty}^{v_0} \int_{-\infty}^{\omega_j(v_j)} f_j(c_j,v_j)dc_jdv_j \prod_{k\in\mathcal{N}\setminus\{i\}\setminus I} \mu_k(D^c_k),$$

and the probability of $v_0 < \tilde{v}_i \leq v$ is

$$\sum_{I\subseteq\mathcal{N}\setminus\{i\},I\neq\emptyset} \{\prod_{j\in I} \int_{-\infty}^{v} \int_{-\infty}^{\omega_j(v_j)} f_j(c_j,v_j)dc_jdv_j - \prod_{j\in I} \int_{-\infty}^{v_0} \int_{-\infty}^{\omega_j(v_j)} f_j(c_j,v_j)dc_jdv_j\} \prod_{k\in\mathcal{N}\setminus\{i\}\setminus I} \mu_k(D^c_k),$$

which has a density to be denoted by $\tilde{g}_i(v;\omega_{-i})$. We have $\tilde{G}_i(v;\omega_{-i}) = 0$ if $v < v_0$ and $\tilde{G}_i(v;\omega_{-i}) = 1$ if $v > \max_{j\neq i}\{\overline{v}_j\}$, and $\tilde{g}_i(v;\omega_{-i}) = 0$ if $v < v_0$ or $v > \max_{j\neq i}\{\overline{v}_j\}$. Note that $\tilde{G}_i(v_0;\omega_{-i})$ can be strictly positive since v_0 can be a mass point .

Define

$$S_i(t_i;\omega_{-i}) = \tilde{G}_i(v_0;\omega_{-i})\max(v_i - v_0, 0) + \int_{v_0}^{v_i}(v_i - v)\tilde{g}_i(v;\omega_{-i})dv - c_i, \forall t_i = (c_i, v_i) \in \Gamma_p^i(\omega_i).$$

Let $\tilde{v}_i^* = \tilde{v}_i(\mathbf{t}_{-i};\omega_{-i}^*)$. Then $\tilde{v}_i^*$ has a density function of $\tilde{g}_i(v;\omega_{-i}^*)$ for $v > v_0$. In addition, $Pr(\tilde{v}_i^* = v_0) = \tilde{G}_i(v_0;\omega_{-i}^*)$. Thus, $S_i((c_i, v_i);\omega_{-i}^*)$ can be interpreted as the marginal contribution of bidder i of type (c_i, v_i) to $S(\omega^*)$. Note that $S_i((c_i, v_i);\omega_{-i}^*)$ is continuous on T_i due to the existence of density $\tilde{g}_i(v;\omega_{-i}^*)$.

Using the above new notations, $S(\omega^*)$ can be alternatively written as

$$S(\omega^*) = \{E\tilde{v}_i^* - \sum_{j\neq i}\int_{t_j=(c_j,v_j)\in\Gamma_p^j(\omega_j^*)} c_j f_j(t_j)dt_j\} + \int_{\underline{v}_i}^{\overline{v}_i}\int_{\underline{c}_i}^{\omega_i^*(v_i)} S_i(t_i;\omega_{-i}^*)f_i(t_i)dt_i.$$

Since $\omega^* \in argmax_{\omega\in\Omega}S(\omega)$, we must have

$$\omega_i^*(\cdot) \in argmax_{w_i(\cdot)\in\Omega_i}\int_{\underline{v}_i}^{\overline{v}_i}\int_{\underline{c}_i}^{w_i(v_i)} S_i(t_i;\omega_{-i}^*)f_i(t_i)dt_i.$$

Note that $S_i((c_i, v_i);\omega_{-i}^*)$ strictly decreases with c_i and weakly increases with v_i. We thus can define the following new function $\tilde{\omega}_i^*(\cdot)$ for bidder i. For any $v_i \in [\underline{v}_i, \overline{v}_i]$, if $S_i((\underline{c}_i, v_i);\omega_{-i}^*) \leq 0$, then $\tilde{\omega}_i^*(v_i) = \underline{c}_i$; if $S_i((\overline{c}_i, v_i);\omega_{-i}^*) \geq 0$, then $\tilde{\omega}_i^*(v_i) = \overline{c}_i$; otherwise, $\tilde{\omega}_i^*(v_i)$ is defined by $S_i((\tilde{\omega}_i^*(v_i), v_i);\omega_{-i}^*) = 0$. Clearly, $\tilde{\omega}_i^*(\cdot)$ is well defined and increasing. If $\tilde{\omega}_i^*(\cdot) \equiv \underline{c}_i$ or $\tilde{\omega}_i^*(\cdot) \equiv \overline{c}_i$, it then belongs to Ω_i. Otherwise, there must exist an interval $[\tilde{v}_\ell^{*i}, \tilde{v}_u^{*i}]$ on which $S_i((\tilde{\omega}_i^*(v_i), v_i);\omega_{-i}^*) = 0$. In addition, $\tilde{\omega}_i^*(\cdot) \equiv \underline{c}_i$ if $v_i \in [\underline{v}_i, \tilde{v}_\ell^{*i})$ and $\tilde{\omega}_i^*(\cdot) \equiv \overline{c}_i$ if $v_i \in (\tilde{v}_u^{*i}, \overline{v}_i]$. In this case, we must have $\tilde{v}_\ell^{*i} \geq v_0$. Therefore, we have

$$S_i((\tilde{\omega}_i^*(v_i), v_i);\omega_{-i}^*) = \tilde{G}_i(v_0;\omega_{-i}^*)(v_i - v_0) + \int_{v_0}^{v_i}(v_i - v)\tilde{g}_i(v;\omega_{-i}^*)dv - \tilde{\omega}_i^*(v_i) \equiv 0, \forall v_i \in [\tilde{v}_\ell^{*i}, \tilde{v}_u^{*i}].$$

Differentiating this equation on both sides leads to $\tilde{\omega}_i^{*'}(v_i) = \tilde{G}_i(v_i;\omega_{-i}^*) \in [0, 1],\ \forall v_i \in [\tilde{v}_\ell^{*i}, \tilde{v}_u^{*i}]$, which confirms the Lipschitz condition for $\tilde{\omega}_i^{*'}(v_i)$. This means that $\tilde{\omega}_i^*(\cdot) \in \Omega_i$.

In all the three cases considered in the previous paragraph, the $\tilde{\omega}_i^*(\cdot)$ belongs to Ω_i. Clearly, $\tilde{\omega}_i^*(\cdot)$ maximizes $\int_{\underline{v}_i}^{\overline{v}_i}\int_{\underline{c}_i}^{w_i(v_i)} S_i(t_i;\omega_{-i}^*)f_i(t_i)dt_i$, and any other curve in Ω_i does not maximize the same objective function. Note that the uniqueness of the optimal $\tilde{\omega}_i^*(\cdot)$ is due to positive density

$f_i(\cdot,\cdot)$. Therefore, we must have $\omega_i^*(\cdot) = \tilde{\omega}_i^*(\cdot)$ at every point by the continuity of the two functions. Thus, the properties of $\tilde{\omega}_i^*(\cdot)$ must hold for $\omega_i^*(\cdot)$.

Define (v_ℓ^i, v_u^i) for any $\omega = (\omega_i) \in \Omega$ as in Section 3. Clearly, we have

$$S_i((\omega_i^*(v_i), v_i); \omega_{-i}^*) = 0, \forall v_i \in [v_\ell^{*i}, v_u^{*i}],\ \ \textit{if}\ \ v_\ell^{*i} < v_u^{*i},$$

$$S_i((\underline{c}_i, \overline{v}_i); \omega_{-i}^*) \leq 0,\ \ \textit{if}\ \ \omega_i^*(v_i) \equiv \underline{c}_i,$$

$$S_i((\underline{v}_i, \overline{c}_i); \omega_{-i}^*) \geq 0,\ \ \textit{if}\ \ \omega_i^*(v_i) \equiv \overline{c}_i.$$

Recall $S_i((c_i, v_i); \omega_{-i}^*)$ strictly decreases with c_i and weakly increases with v_i. Therefore we have the properties in Lemma 4 for the contribution of a type t_i to $S(\omega^*)$. $\square$

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